# EXAM 2
# ECEN 478 - Wireless Communications

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## Problem: Coding for wireless fading channels

A study of simple coding techniques for fast and slow fading channels is proposed. The student is assumed to have a minimal background in *Communication Theory*, e.g., Chapter 3 of Tse & Viswanath 2005. Knowledge of *Coding Theory* is not required, all properties related to the code structure are clearly explained in this problem. We restrict this problem to wireless channels with flat fading, i.e., the transmitted signal bandwidth is small compared to the channel coherence bandwidth, $W \ll B_{coh}$. The system model is depicted in Fig. 1. A block of $k$ information bits $u = (u_1\ u_2\ \ldots\ u_k)$, $u_i \in \{0, 1\}$, is fed to a linear binary encoder. The encoder generates a codeword $c$ of length $n$ bits denoted by $c = (c_1\ c_2\ \ldots\ c_n)$, $c_i \in \{0, 1\}$.

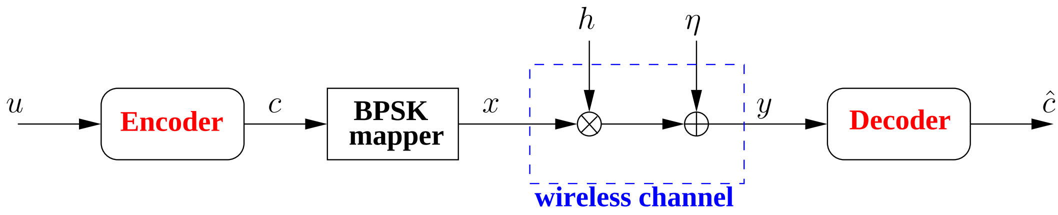


Figure 1: System model for coding over fading channels.

A codeword is computed via the following linear encoding operation

$$c = u \cdot G, \tag{1}$$

where $G$ is a $k \times n$ matrix with binary entries. The set of all possible $2^k$ codewords is an error-correcting code denoted by $C[n, k]$, where $k$ is the code dimension and $n$ is the code length. The coding rate is

$$R_c = \frac{k}{n}, \tag{2}$$

it expresses the number of information bits per coded bit. The matrix $G$ defining the linear operation in (1) is a generator matrix for the code $C$. Of course, all multiplications and additions performed in (1) are modulo 2. The Hamming distance between two codewords $c$ and $c'$ is the number of $c_i$ different from $c'_i$, e.g., $c = (1\ 1\ 0\ 0)$ and $c' = (1\ 0\ 1\ 0)$, then $d_H(c, c') = 2$. The Hamming weight $w_H(c)$ of $c$ is the number of its non-zero binary components, i.e., $w_H(c) = d_H(0, c)$. As an example, if $c = (1\ 0\ 1\ 1)$ then $w_H(c) = 3$. A trivial family of error-correcting codes is the one built by repeating the same bit $n$ times. The repetition code $C[2,1]$ has dimension $k=1$, length $n=2$, rate $R_c = 1/2$, its generator matrix is $G = (1\ 1)$, and the set of codewords is

$$C = \{(0\ 0),\ (1\ 1)\}\,. \tag{3}$$

Another example of a less trivial error-correcting code is the shortened Hamming code $C[6,3]$ (see MacWilliams & Sloane 1977), it has dimension $k=3$, length $n=6$ bits, and rate $R_c = 1/2$. A generator matrix for $C[6,3]$ is

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \tag{4}$$

The list of $2^k = 8$ possible codewords $c = (c_1\ c_2\ c_3\ c_4\ c_5\ c_6)$ of $C[6,3]$ is given in Table 1. The table also gives the information block $u$ for each codeword $c = uG$. The third column indicates the total Hamming weight of $c$. The last two columns indicate the partial Hamming weights defined as

$$w_1(c) = w_H(c_1\ c_2\ c_3), \qquad w_2(c) = w_H(c_4\ c_5\ c_6). \tag{5}$$

| Information bits $u$ | Codeword $c$ | $w_H(c)$ | $w_1(c)$ | $w_2(c)$ |
|---|---|---|---|---|
| 000 | 000 000 | 0 | 0 | 0 |
| 001 | 001 110 | 3 | 1 | 2 |
| 010 | 010 101 | 3 | 1 | 2 |
| 011 | 011 011 | 4 | 2 | 2 |
| 100 | 100 011 | 3 | 1 | 2 |
| 101 | 101 101 | 4 | 2 | 2 |
| 110 | 110 110 | 4 | 2 | 2 |
| 111 | 111 000 | 3 | 3 | 0 |

Table 1: The rate-1/2 shortened Hamming code $C[6,3]$.

The Hamming weight distribution for non-zero codewords of a linear binary code $C[n,k]$ can be compactly summarized in a weight enumerating polynomial

$$A(z) \;=\; \sum_{w>0} A_w\ z^w, \tag{6}$$

where $A_i$ is the number of codewords of weight equal to $w$. For example, the weight enumerator for $C[2,1]$ is $A(z) = z^2$. Also, from Table 1, we find the weigh enumerator for $C[6,3]$ being equal to $A(z) = 4z^3 + 3z^4$. Similarly, the partial weight enumeration is summarized by

$$A(z_1, z_2) \;=\; \sum_{w_1,w_2} A_{w_1 w_2}\ z_1^{w_1}\ z_2^{w_2}, \tag{7}$$

where $A_{w_1 w_2}$ is the number of codewords $c$ with partial Hamming weights equal to $w_1$ and $w_2$ respectively. For $C[2,1]$, we have $A(z_1, z_2) = z_1 z_2$. For $C[6,3]$, we have $A(z_1, z_2) = 3z_1 z_2^2 + 3z_1^2 z_2^2 + z_1^3$. Finally, notice that the two polynomials are related by $A(z) = A(z, z)$.

As illustrated in Fig. 1, a binary codeword $c = (c_1 \ c_2 \ \ldots \ c_n)$ is mapped into a BPSK codeword $x = (x_1 \ x_2 \ \ldots \ x_n)$, where

$$x_i = \Psi(c_i) = \begin{cases} -\sqrt{E_s}, & c_i = 0, \\ \\ +\sqrt{E_s}, & c_i = 1, \end{cases} \tag{8}$$

where $E_s$ is the baseband energy per transmitted symbol $x_i$. In the single antenna case, the wireless channel multiplies the symbol by a fading $h_i$ before the addition of a gaussian noise $\eta_i$. The channel model is

$$y_i = h_i x_i + \eta_i, \qquad y_i \in \mathbb{C}, \tag{9}$$

where $i = 1 \ldots n$, the fading coefficients $h_i$ are distributed as $\mathcal{CN}(0,1)$, and the noise samples are independent and distributed as $\mathcal{CN}(0, 2\sigma^2)$. The wireless channel model in (9) can be written in vector notation as $y = h \odot x + \eta$, where $\odot$ denotes component-wise multiplication.

Based on the observation $y = (y_1 \ y_2 \ \ldots \ y_n)$, the decoder is supposed to decide which is the most likely transmitted codeword $\hat{x}$ or equivalently $\hat{c}$. The vector $h = (h_1 \ h_2 \ \ldots \ h_n)$ of fading coefficients is assumed to be perfectly known by the decoder. The likelihood of a codeword $x$ given $h$ is

$$p(y|x,h) = \frac{1}{(2\pi\sigma^2)^n} \exp(-\frac{\|y - h \odot x\|^2}{2\sigma^2}), \tag{10}$$

where $\|y - h \odot x\|^2 = \sum_{i=1}^{n} |y_i - h_i x_i|^2$. For a given observation $y$ and a given state vector $h$, the decoder finds the most likely codeword $\hat{x}$ that minimizes the Euclidean metric,

$$\hat{x} = \arg \min_{x \in C} \|y - h \odot x\|^2. \qquad (ML\ decoding) \tag{11}$$

In the sequel, we assume that the zero-codeword $\underline{0} = (0 \ 0 \ \ldots \ 0)$ has been transmitted by the encoder. After BPSK mapping, the transmitted codeword is

$$x^0 = \Psi(\underline{0}) = (-\sqrt{E_s} \ \ -\sqrt{E_s} \ \ \ldots \ \ -\sqrt{E_s}). \tag{12}$$

The word error probability is defined as

$$P_{ew} = P(\hat{x} \neq x^0) = P(\hat{c} \neq \underline{0}). \tag{13}$$

Using the Union bound, $P_{ew}$ can be upper-bounded as follows

$$P_{ew} \leq \sum_{c \neq \underline{0}} P(\underline{0} \to c) = \sum_{x \neq x^0} P(x^0 \to x). \tag{14}$$

The pairwise error probability is derived from the conditional pairwise error probability after averaging over the fading vector $h$,

$$P(\underline{0} \to c) = P(x^0 \to x) = \int_h p(h) \ P(x^0 \to x \ |h) \ dh. \tag{15}$$

# Part I: Single antenna coding for fast fading

The channel state vector $h = (h_1 \ h_2 \ \dots \ h_n)$ includes $n$ independent and identically distributed coefficients $h_i \sim \mathcal{CN}(0,1)$.

**I.1)** As stated above, the channel state vector $h$ is known by the decoder. Is it a coherent or a non-coherent receiver?

**I.2)** Let $\alpha_i = |h_i|$. What is the probability density function of $\alpha_i$?

**I.3)** Briefly explain why the conditional pairwise error probability $P(\underline{0} \to c\,|h) = P(x^0 \to x\,|h)$ is equal to

$$P(\underline{0} \to c\,|h) \ = \ P(\|y - h \odot x\|^2 < \|y - h \odot x^0\|^2), \tag{16}$$

where $x = \Psi(c)$ and $y = h \odot x^0 + \eta$.

**I.4)** For a fixed fading vector $h$, briefly explain why the conditional pairwise error probability can be expressed using the gaussian tail function as

$$P(\underline{0} \to c\,|h) \ = \ Q\left(\frac{\|h \odot x - h \odot x^0\|}{2\sigma}\right). \tag{17}$$

**I.5)** Take $\sigma^2 = N_0/2$. Prove that

$$P(\underline{0} \to c\,|h) \ = \ Q\left(\sqrt{\frac{\sum_{i=1}^{n} |h_i|^2 (x_i + \sqrt{E_s})^2}{2N_0}}\right), \tag{18}$$

where $x_i = \Psi(c_i)$.

**I.6)** A closed-form expression for $P(\underline{0} \to c)$ is known after integrating (18) over $h$. Nevertheless, a better physical interpretation can be obtained by using $Q(x) \leq \frac{1}{2} e^{-x^2/2}$ and then integrating over the i.i.d. Rayleigh distributed variables. It can be shown that

$$P(\underline{0} \to c) \ \leq \ \frac{1}{2} \prod_{i=1}^{n} \frac{1}{1 + \frac{(x_i + \sqrt{E_s})^2}{4N_0}} \tag{19}$$

Prove that

$$P(\underline{0} \to c) \ \leq \ \frac{1}{2} \prod_{i=1}^{n} \frac{1}{1 + c_i \gamma} \ \approx \ \frac{0.5}{\gamma^{w_H(c)}}, \tag{20}$$

where $\gamma = E_s/N_0$ is the signal-to-noise ratio. The right-hand approximation in (20) is valid for large $\gamma$.

**I.7)** At high SNR ($\gamma \gg 1$), using (14) and (20), prove that

$$P_{ew} \lesssim \frac{1}{2} A(\frac{1}{\gamma}), \tag{21}$$

where $A(z)$ is the weight enumerating polynomial of the code $C[n, k]$.

**I.8)** On a fast fading channel, after examination of the minimum exponent of the SNR $\gamma$ in the word error rate $P_{ew}$, what is the diversity order achieved by the rate-1/2 repetition code $C[2, 1]$? What is the diversity order of the rate-1/2 shortened Hamming code $C[6, 3]$? The word error rates for $C[2, 1]$ and $C[6, 3]$ are plotted in Fig. 2 in the Appendix.

# Part II: Single antenna coding for slow fading

In this section, the coefficients $h_i$ are not independent, they may stay invariant within a codeword. The channel state vector $h$ is now divided into two parts,

$$h = (h_1 \ \ldots \ h_{n/2} \mid h_{n/2+1} \ \ldots \ h_n). \tag{22}$$

We have $h_1 = h_2 = \ldots = h_{n/2} = g_1$ and $h_{n/2+1} = \ldots = h_n = g_2$, where $g_1$ and $g_2$ are independent and identically distributed coefficients, $g_i \sim \mathcal{CN}(0,1)$. In the case of $C[6,3]$ $(n = 6)$, we have $h = (g_1, g_1, g_1, g_2, g_2, g_2)$.

**II.1)** The fading vector of length $n$ is built from two independent complex gaussian fadings $g_1$ and $g_2$ as described above. What is the maximal diversity order achievable on this slow fading channel? A channel also known as a non-ergodic fading channel.

**II.2)** In a way similar to I.5, prove that

$$P(\underline{0} \to c \ |h) \ = \ Q\left(\sqrt{\frac{|g_1|^2 \sum_{i=1}^{n/2}(x_i + \sqrt{E_s})^2 + |g_2|^2 \sum_{i=n/2+1}^{n}(x_i + \sqrt{E_s})^2}{2N_0}}\right), \tag{23}$$

**II.3)** In a way similar to I.6, prove that

$$P(\underline{0} \to c) \ \leq \ \frac{0.5}{(1 + w_1(c)\gamma) \ \ (1 + w_2(c)\gamma)}. \tag{24}$$

Hence, show that a code $C[n, k]$ attains maximal diversity if and only if $A(z_1, z_2)$ is divisible by $z_1 z_2$.

**II.4)** When $C[n, k]$ is full diversity, show that

$$P_{ew} \ \lesssim \ \frac{1}{2} \sum_{w_1, w_2} \frac{A_{w_1,w_2}}{w_1 \cdot w_2 \cdot \gamma^2}. \tag{25}$$

**II.5)** The code $C[6, 3]$ given in Table 1 has diversity 1 (no diversity) due to the last codeword in the table where $w_1 = 3$ but $w_2 = 0$. We suggest to build a new version denoted by $\pi C[6, 3]$. If $c = (c_1 \ c_2 \ c_3 \ c_4 \ c_5 \ c_6) \in C$ then $\pi c = (c_6 \ c_2 \ c_3 \ c_4 \ c_5 \ c_1) \in \pi C$, i.e., $\pi C[6, 3]$ is obtained by exchanging bit $c_1$ and bit $c_6$ in the second column of Table 1. Find the weight enumerator polynomial $A(z_1, z_2)$ of the new code version $\pi C[6, 3]$.

**II.6)** Show that $\pi C[6, 3]$ attains the full diversity guaranteed by the slow fading channel defined above.

**II.7)** At high SNR ($\gamma \gg 1$), prove that the word error rate of $\pi C[6, 3]$ can be upper bounded as follows

$$P_{ew} \lesssim \frac{35/24}{\gamma^2}, \tag{26}$$

The word error rates for $C[6, 3]$ and $\pi C[6, 3]$ on a slow fading channel are plotted in Fig. 3 in the Appendix.

**Extension:** The reader can study the coding of a $2 \times 1$ MIMO channel using the error-correcting codes proposed above. The rank criterion can be easily applied to $C[6, 3]$ or its permuted version to check if full transmit diversity is achieved.

# Appendix: Performance of error-correcting codes

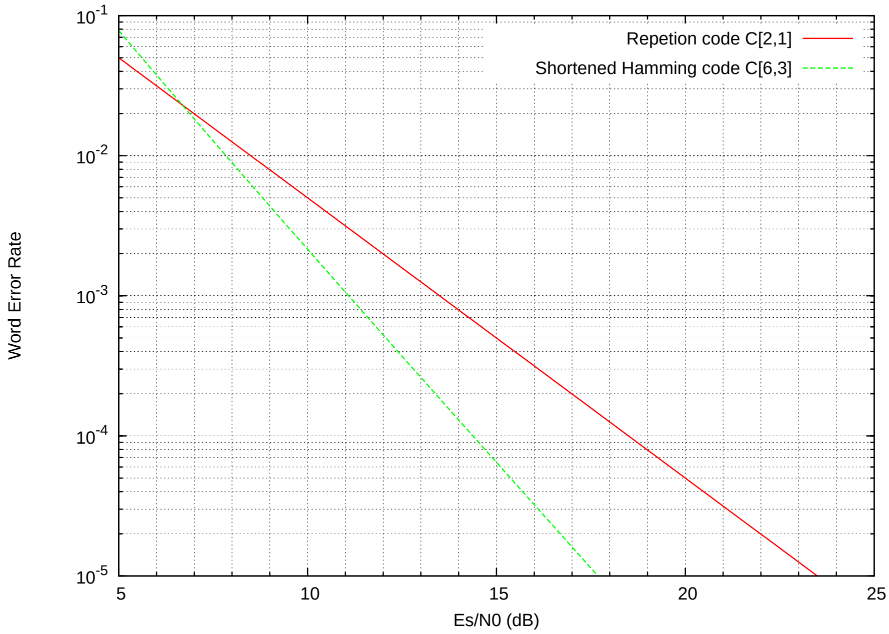


Figure 2: Performance over a fast fading channel, $P_{ew}$ is approximated by $\frac{1}{2}A(\gamma^{-1})$.

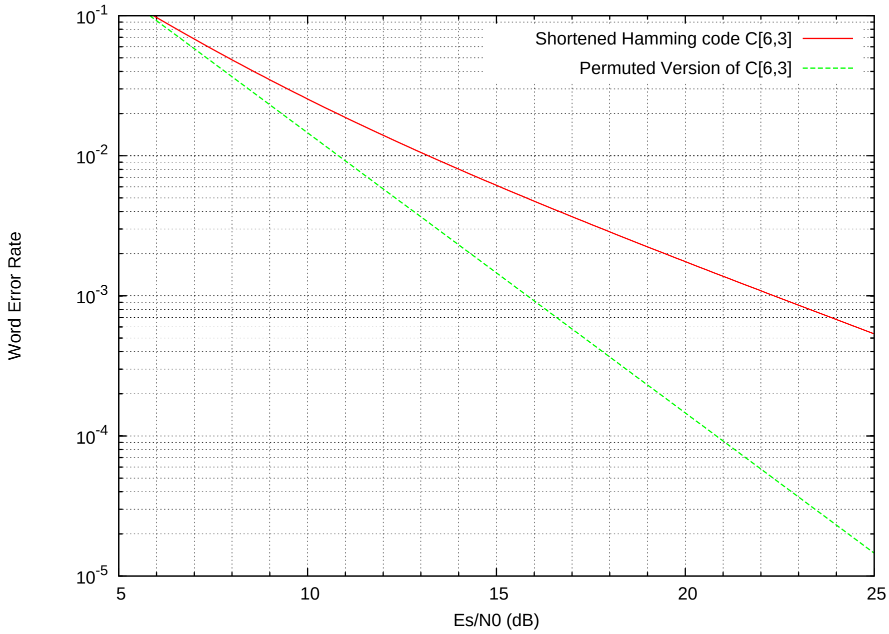


Figure 3: Performance over a slow fading channel, $P_{ew}$ is derived from the weight enumerator $A(z_1, z_2)$.

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