EXAM 1
ECEN 478 - Wireless Communications

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Problem I: Warm-up questions in wireless communications theory

I.1 Consider a baseband channel model with a fading process $\alpha(t)$ assumed to be stationary in time. The channel output at time instant t is

$$y(t) = \alpha(t) \times x(t), \tag{1}$$

where $x(t)$ is the channel input. Additive noise is not considered in this problem. The power spectral density $S_\alpha(f)$ of $\alpha(t)$ is non-zero inside the interval $[-B_d/2 \ldots + B_d/2]$.

(I.1.a) Is it a frequency-selective channel ? Guess its coherence bandwidth.
(I.1.b) Is it a time-selective channel ? Guess its coherence time.
(I.1.c) The frequency bandwidth of the transmitted signal $x(t)$ is W. Determine the bandwidth of the received signal $y(t)$.

I.2 In a typical urban area, the multipath delay spread of a wireless channel is $T_d \approx 5\mu sec$. The bandwidth W of the transmitted signal is $W = 1MHz$.

(I.2.a) Is it a frequency-selective channel ? Determine its coherence bandwidth.
(I.2.b) Given that the receiver is operating at a sampling period $1/W$, compute the number of channel paths L that can be distinguished.
(I.2.c) Consider the special case $L = 2$ where the channel impulse response (LTV) is

$$h(\tau, t) = a_1(t)\delta(\tau) + a_2(t)\delta(\tau - \tau_0). \tag{2}$$

The path coefficients $a_1(t)$ and $a_2(t)$ are two independent complex gaussian (circularly symmetric) stationary processes with zero mean and an identical correlation function $R_a(\Delta t) = E[a_i(t + \Delta t)a_i^*(t)]$, $i = 1, 2$. Compute the scattering function $S_h(\tau, \nu)$ via a double Fourier integral starting from the spaced-frequency spaced-time correlation function $\Phi_H(\Delta f, \Delta t)$.
Express $S_h(\tau, \nu)$ using $S_a(\nu) = \mathcal{F}[R_a(\Delta t)]$. Find an approximate value for the Doppler bandwidth if the time support of $R_a(\Delta t)$ is equal to 0.01 sec.

Problem II: Wireless channels with correlated fadings and unequal power paths

A digital communication system over fading channels has an L-order diversity via an information replicating procedure such as time interleaving diversity, frequency hopping diversity, spatial receive diversity, etc. The transmitted BPSK symbol takes two values $x = \pm A$. The discrete-time baseband model for a coherent detection is given by the following real model ($L = 2$ for simplicity)

$$\begin{cases} y_1 = \alpha_1 x + \eta_1, \\ y_2 = \alpha_2 x + \eta_2, \end{cases} \tag{3}$$

where $\alpha_i = |h_i|$, $i = 1, 2$, h_1 and h_2 are jointly complex gaussian circularly symmetric random variables with zero mean, i.e., α_i is Rayleigh distributed. The AWG noise variance per real component is σ^2, i.e., $\eta_i \sim \mathcal{N}(0, \sigma^2)$.
After maximum-ratio combining, the decision variable becomes

$$\alpha_1 y_1 + \alpha_2 y_2 = (\alpha_1^2 + \alpha_2^2)x + \eta = (|h_1|^2 + |h_2|^2)x + \eta, \tag{4}$$

where $\eta = \alpha_1\eta_1 + \alpha_2\eta_2$. It has been proven (see Chapter 1 of ECEN 478) that the conditional bit error probability is given by

$$P_e(\alpha_1, \alpha_2) = Q\left(\sqrt{(\alpha_1^2 + \alpha_2^2)\frac{A^2}{\sigma^2}}\right) = Q\left(\sqrt{2(|h_1|^2 + |h_2|^2)\gamma}\right). \tag{5}$$

Here, $Q(x)$ denotes the gaussian tail function. The parameter $\gamma = E_b/N_0$ is the signal-to-noise ratio per bit at the transmitter side, assuming a unit variance for h_1 and h_2. In order to study a more universal channel, we include γ in the variance of the two fading coefficients and we rewrite the conditional error probability as

$$P_e(h) = Q\left(\sqrt{2(|h_1|^2 + |h_2|^2)}\right) = Q\left(\sqrt{2h^\dagger h}\right), \tag{6}$$

where $E[|h_1|^2] = \gamma_1$, $E[|h_2|^2] = \gamma_2$, and $E[h_1 h_2^*] = \beta \times \sqrt{\gamma_1\gamma_2}$ ($\beta \in \mathbb{R}$). As usual, the final average error probability is obtained by averaging over the fading vector h,

$$P_e = \int_h P_e(h)p(h)dh. \tag{7}$$

Let Σ be the covariance matrix of the fading vector h,

$$\Sigma = E[hh^\dagger] = \begin{bmatrix} \gamma_1 & \beta \times \sqrt{\gamma_1\gamma_2} \\ \beta \times \sqrt{\gamma_1\gamma_2} & \gamma_2 \end{bmatrix}. \tag{8}$$

Denote by λ_1 and λ_2 the two eigen values of Σ. Then, after integration and some algebra (e.g., see Venugopal Veeravalli, 2001), the error probability in (7) becomes

$$P_e = \frac{1}{2}\sum_{\ell=1}^{L} \rho_\ell \left[1 - \sqrt{\frac{\lambda_\ell}{1 + \lambda_\ell}}\right], \tag{9}$$

where $\rho_\ell = \prod_{i \neq \ell} \left(1 - \frac{\lambda_i}{\lambda_\ell}\right)^{-1}$.

II.1 Unequal path gains and independent fadings

In this section we assume that $\beta = 0$ and $\gamma_1 \neq \gamma_2$.

II.1.a Find the eigen values of Σ.

II.1.b Write the expression of P_e versus γ_1 and γ_2.

II.1.c At high SNR, $\gamma_1 \gg 1$ and $\gamma_2 \gg 1$, prove that

$$P_e \approx \frac{3}{16} \frac{1}{\gamma_1 \gamma_2}. \tag{10}$$

II.2 Equal path gains and correlated fadings

In this section we assume that $\beta \neq 0$ and $\gamma = \gamma_1 = \gamma_2$.

II.2.a Find the eigen values of Σ.

II.2.b Write the expression of P_e versus β and γ.

II.2.c At high SNR, $\gamma \gg 1$, prove that

$$P_e \approx \frac{3}{16} \frac{1}{(1-\beta^2)\gamma^2}. \tag{11}$$

III Conclusion

From the above results, explain why the lowest error probability is obtained with independent fadings and channel paths of equal energy.

.Good Luck.

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