the average frequency distribution

$$H_{\rm avg}(s) := (1/N)\sum_{i=1}^{N} H(s, i).$$

The behavior of the distributions for different values of K is discussed in Fig. 4. For $M = 9$ and $K = 12, 16, 20, 24$ the average frequency distribution $H_{\rm avg}(s)$ (solid lines) and the approximation $H_{\rm app}(s)$ (dashed lines) are compared to the Maxwell–Boltzmann distribution $H_{\rm M\text{-}B}(s)$ (dash-dotted lines). Here, the parameter λ is chosen so that the entropy is equal to $K/8$. Even for low K the approximation $H_{\rm app}(s)$ is very close to the true average frequency distribution $H_{\rm avg}(s)$. The approximation improves as K increases. Unfortunately, the Maxwell–Boltzmann distribution $H_{\rm M\text{-}B}(s)$ does not provide a good estimate of $H_{\rm app}(s)$. Shells with low index occur less often than expected from the optimal entropy–power tradeoff.

Finally, in Table II the average energy σ_a^2 of the signal points in V.34 are summarized. For a symbol rate of 3200 Hz, the true average energy σ_a^2 (cf. (2)), the approximate energy $\sigma_{a,\,\rm app}^2$ based on $H_{\rm app}(s)$, and the energy $\sigma_{a,\,\rm M\text{-}B}^2$ derived from the Maxwell–Boltzmann distribution are given for all possible data rates and associated mapping parameters K, M, and q [5, Table 10, expanded]. The underlying signal constellation is specified in [5, Fig. 5]. Again, the exact calculation and the approximation are very close. Obviously, the energies derived from the Maxwell–Boltzmann distribution underestimate the actual energies as they are lower bounds. The approximation (13) provides much better results.

IV. Conclusions

In this correspondence, a simple but general method for the calculation of the frequencies of the shells in shell-mapping schemes was derived. As an example, the method was shown in detail for the shell-mapping scheme specified for the international telephone-line modem standard ITU Recommendation V.34. The method starts with partial histograms that give the number of occurrences of shells within all possible combinations of n-tuples of shells with some fixed total cost. These histograms can be calculated easily using the generating functions that are needed in the encoder in any case. Then, the shell-mapping encoder is run with a specific input, namely, the maximum K-tuple. To each step of the encoding procedure a partial histogram can be assigned. Summing up these parts yields the final histograms. Thus the calculation has approximately the same complexity as the mapping encoder itself. With the knowledge of the frequencies of shells, the exact average transmit power can be calculated. Numerical examples are given for V.34.

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References

[1] I. N. Bronstein and K. A. Semendjajew, *Taschenbuch der Mathematik.* Thün, Frankfurt/Main, Germany: Verlag Harri Deutsch, 1987, in German.

[2] M. V. Eyuboğlu, G. D. Forney, P. Dong, and G. Long, “Advanced modulation techniques for V.fast,” *Europ. Trans. Telecommun.*, vol. 4, pp. 243–256, May/June 1993.

[3] G. D. Forney and L.-F. Wei, “Multidimensional constellations—Part I: Introduction, figures of merit, and generalized cross constellations,” *IEEE J. Select. Areas Commun.*, vol. 7, pp. 877–892, 1989.

[4] P. Fortier, A. Ruiz, and J. M. Cioffi, “Multidimensional signal sets through the shell construction for parallel channels,” *IEEE Trans. Commun.*, vol. 40, pp. 500–512, Mar. 1992.

[5] International Telecommunication Union (ITU), Std. V.34, “A modem operating at data signalling rates of up to 28800 bit/s for use on the general switched telephone network and on leased point-to-point 2-wire telephone-type circuits,” Sept. 1994.

[6] A. K. Khandani and P. Kabal, “Shaping multidimensional signal spaces—Part I: Optimum shaping, shell mapping; Part II: Shell-addressed constellations,” *IEEE Trans. Inform. Theory*, vol. 39, pp. 1799–1819, Nov. 1993.

[7] F. R. Kschischang and S. Pasupathy, “Optimal nonuniform signaling for Gaussian channels,” *IEEE Trans. Inform. Theory*, vol. 39, pp. 913–929, May 1993.

[8] G. R. Lang and F. M. Longstaff, “A Leech lattice modem,” *IEEE J. Select. Areas Commun.*, vol. 7, pp. 986–973, Aug. 1989.

[9] R. Laroia, N. Farvardin, and S. A. Tretter, “On optimal shaping of multidimensional constellations,” *IEEE Trans. Inform. Theory*, vol. 40, pp. 1044–1056, July 1994.

A Universal Lattice Code Decoder for Fading Channels

Emanuele Viterbo, *Member, IEEE*, and Joseph Boutros, *Member, IEEE*

***Abstract*—We present a maximum-likelihood decoding algorithm for an arbitrary lattice code when used over an independent fading channel with perfect channel state information at the receiver. The decoder is based on a bounded distance search among the lattice points falling inside a sphere centered at the receved point. By judicious choice of the decoding radius we show that this decoder can be practically used to decode lattice codes of dimension up to 32 in a fading environement.**

***Index Terms*— Maximum-likelihood decoding, modulation, lattices, wireless channel.**

I. Introduction

Lattice codes are used in digital transmission as high-rate signal constellations. They are obtained by carving a finite number of points from an n-dimensional lattice in the Euclidean space $\boldsymbol{R}^n$. For the basic notations in lattice theory the reader can refer to [1]. Maximum-likelihood (ML) decoding of a lattice code used over an additive white Gaussian noise (AWGN) channel is equivalent to finding the closest lattice point to the received point. Many very efficient algorithms are now available for ML decoding some well-known root lattices [1]. Several Leech lattice decoders have been proposed with an ever-improving efficiency; a review of these decoders can be found in [2]. The above algorithms are strictly dependent on the special structure of the lattice being decoded (e.g., its being a binary lattice). Other algorithms for general nearest neighbor encoding in vector quantization are valid for any unstructured codebook. They do not take full advantage of the lattice structure which is useful for large

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E. Viterbo is with AT&T Shannon Laboratories, Florham Park, NJ 07932 USA, on leave from Politecnico di Torino, I-10129 Torino, Italy.

J. Boutros is with Ecole Nationale Supérieure des Télécommunications, 75634 Paris, France.

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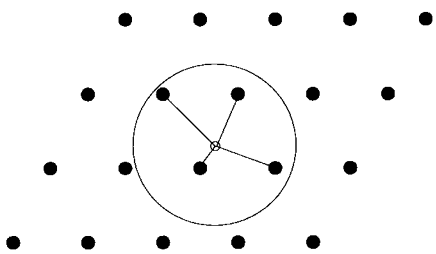

Fig. 1. Geometrical representation of the sphere decoding algorithm.

bit-rate applications [3]. As we will see in the following, when dealing with lattice codes for the fading channel we are faced with the problem of decoding a totally arbitrary lattice given its generator matrix.

Recent work on multidimensional modulation schemes for the fading channel show how to construct lattice codes well adapted for such a channel [4]. These lattice codes are effective because they present a high *modulation diversity* L, i.e., any two code vectors always differ in at least L coordinates.

In the case of independent fading channels, with perfect channel state information (CSI) given to the receiver, ML decoding requires the minimization of the following metric:

$$m(\boldsymbol{x} \mid \boldsymbol{r}, \boldsymbol{\alpha}) = \sum_{i=1}^{n} |r_i - \alpha_i x_i|^2 \tag{1}$$

where $\boldsymbol{r} = \boldsymbol{\alpha} * \boldsymbol{x} + \boldsymbol{n}$ is the received vector. The noise vector $\boldsymbol{n} = (n_1, n_2, \cdots, n_n)$ has real, Gaussian distributed independent random variable components, with zero mean and N_0 variance. The random independent fading coefficients $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ have unit second moment and $*$ represents the component-wise product. $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ is one of the transmitted lattice code points.

The lattice points can be written as the set $\{\boldsymbol{x} = \boldsymbol{u}M\}$, where M is the lattice generator matrix corresponding to the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ and $\boldsymbol{u} = (u_1, \cdots, u_n)$ is the integer component vector to which the information bits are easily mapped. Signal demodulation is assumed to be coherent, so that the fading coefficients can be modeled, after phase elimination, as real random variables with a Rayleigh distribution. In practice, a component interleaver is needed to obtain the desired independence of the fading coefficients α_i.

The algorithm proposed in this correspondence enables to find the closest point of the lattice constellation in terms of metric (1) and practically solves the docoding problem at least for dimensions up to 32.

II. The Sphere-Decoder Algorithm

We consider first the Gaussian channel case so that we can assume $\alpha_i = 1$, $i = 1, \cdots, n$. To the authors knowledge the following algorithm was first presented in [6] and further analyzed in [7] and [8]. We report here a simple derivation of the algorithm which can then be easily implemented using the flow chart of Fig. 2.

The lattice decoding algorithm searches through the points of the lattice Λ which are found inside a sphere of given radius $\sqrt{C}$ centered at the received point, as shown in Fig. 1. This guarantees that only the lattice points within the square distance C from the received point are considered in the metric minimization.

In the following, it is useful to think of the lattice Λ as the result of a linear transformation, defined by the matrix $M : \boldsymbol{R}^n \to \boldsymbol{R}^n$, when applied to the cubic lattice $\boldsymbol{Z}^n$.

The problem to solve is the following:

$$\min_{\boldsymbol{x}\in\Lambda} \|\boldsymbol{r} - \boldsymbol{x}\| = \min_{\boldsymbol{w}\in \boldsymbol{r}-\Lambda} \|\boldsymbol{w}\| \tag{2}$$

that is, we search for the shortest vector $\boldsymbol{w}$ in the translated lattice $\boldsymbol{r} - \Lambda$ in the n-dimensional Euclidean space $\boldsymbol{R}^n$.

We write $\boldsymbol{x} = \boldsymbol{u}M$ with $\boldsymbol{u} \in \boldsymbol{Z}^n$, $\boldsymbol{r} = \boldsymbol{\rho}M$ with $\boldsymbol{\rho} = (\rho_1, \cdots, \rho_n) \in \boldsymbol{R}^n$, and $\boldsymbol{w} = \boldsymbol{\xi}M$ with $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n$. Note that $\boldsymbol{\rho}$ and $\boldsymbol{\xi}$ are real vectors. Then we have $\boldsymbol{w} = \sum_{i=1}^n \xi_i \boldsymbol{v}_i$, where $\xi_i = \rho_i - u_i$, $i = 1, \cdots, n$ define the translated coordinate axes in the space of the integer component vectors $\boldsymbol{u}$ of the cubic lattice $\boldsymbol{Z}^n$.

The sphere of square radius C and centered at the received point is transformed into an ellipsoid centered at the origin of the new coordinate system defined by $\boldsymbol{\xi}$

$$\|\boldsymbol{w}\|^2 = Q(\boldsymbol{\xi}) = \boldsymbol{\xi} M M^T \boldsymbol{\xi}^T = \boldsymbol{\xi} G \boldsymbol{\xi}^T = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\xi_i\xi_j \le C. \tag{3}$$

Cholesky's factorization of the Gram matrix $G = MM^T$ yields $G = R^T R$, where R is an upper triangular matrix. Then

$$Q(\boldsymbol{\xi}) = \boldsymbol{\xi} R^T R \boldsymbol{\xi}^T = \|R\boldsymbol{\xi}^T\|^2 = \sum_{i=1}^{n}\left(r_{ii}\xi_i + \sum_{j=i+1}^{n} r_{ij}\xi_j\right)^2 \le C. \tag{4}$$

Substituting $q_{ii} = r_{ii}^2$ for $i = 1, \cdots, n$ and $q_{ij} = r_{ij}/r_{ii}$ for $i = 1, \cdots n$, $j = i+1, \cdots, n$, we can write

$$Q(\boldsymbol{\xi}) = \sum_{i=1}^{n} q_{ii}\left(\xi_i + \sum_{j=i+1}^{n} q_{ij}\xi_j\right)^2 \le C. \tag{5}$$

Starting from ξ_n and working backwards, we find the equations of the border of the ellipsoid. The corresponding ranges for the integer components u_n and u_{n-1} are

$$\left\lceil -\sqrt{\frac{C}{q_{nn}}} + \rho_n \right\rceil \le u_n \le \left\lfloor \sqrt{\frac{C}{q_{nn}}} + \rho_n \right\rfloor$$

$$\left\lceil -\sqrt{\frac{C - q_{nn}\xi_n^2}{q_{n-1,n-1}}} + \rho_{n-1} + q_{n-1,n}\xi_n \right\rceil \le u_{n-1} \le \left\lfloor \sqrt{\frac{C - q_{nn}\xi_n^2}{q_{n-1,n-1}}} + \rho_{n-1} + q_{n-1,n}\xi_n \right\rfloor$$

where $\lceil x \rceil$ is the smallest integer greater than x and $\lfloor x \rfloor$ is the greatest integer smaller than x. For the ith integer component we have

$$\left\lceil -\sqrt{\frac{1}{q_{ii}}\left(C - \sum_{l=i+1}^{n} q_{ll}\left(\xi_l + \sum_{j=l+1}^{n} q_{lj}\xi_j\right)^2\right)} + \rho_i + \sum_{j=i+1}^{n} q_{ij}\xi_j \right\rceil \le u_i \le \left\lfloor \sqrt{\frac{1}{q_{ii}}\left(C - \sum_{l=i+1}^{n} q_{ll}\left(\xi_l + \sum_{j=l+1}^{n} q_{lj}\xi_j\right)^2\right)} + \rho_i + \sum_{j=i+1}^{n} q_{ij}\xi_j \right\rfloor. \tag{6}$$

The search algorithm proceeds very much like a mixed-radix counter on the digits u_i, with the addition that the bounds change whenever there is a carry operation from one digit to the next. In practice, the bounds can be updated recursively by using the following equations:

$$S_i = S_i(\xi_{i+1}, \cdots, \xi_n) = \rho_i + \sum_{l=i+1}^{n} q_{il}\xi_l$$

$$T_{i-1} = T_{i-1}(\xi_i, \cdots, \xi_n) = C - \sum_{l=i}^{n} q_{ll}\left(\xi_l + \sum_{j=l+1}^{n} q_{lj}\xi_j\right)^2 = T_i - q_{ii}(S_i - u_i)^2.$$

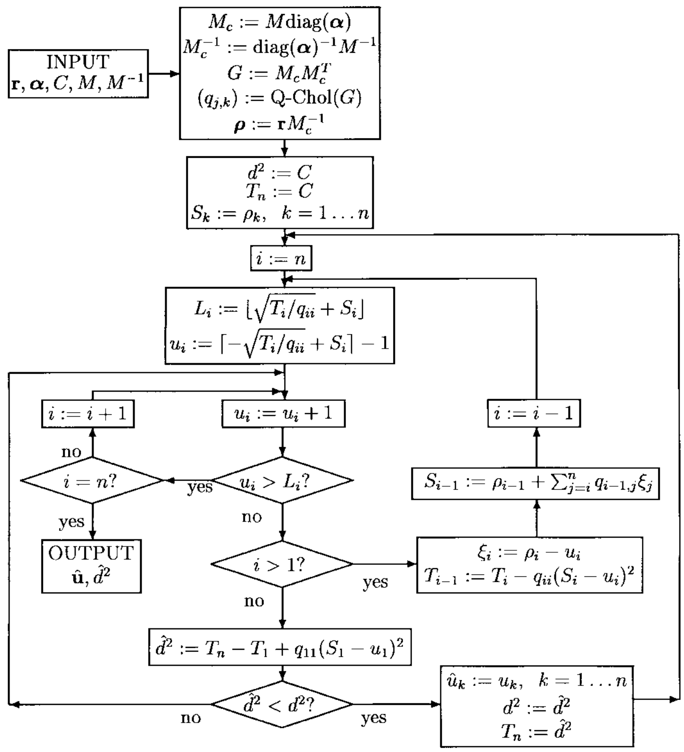

Fig. 2. Flowchart of the lattice decoding algorithm with fading. The function Q-Chol computes the q_{ij} terms of (5).

When a vector inside the sphere is found, its square distance from the center (the received point) is given by

$$\hat{d}^2 = C - T_1 + q_{11}(S_1 - u_1)^2.$$

This value is compared to the minimum square distance d^2 (initially set equal to C) found so far in the search. If it is smaller then we have a new candidate closest point and the search continues like this until all the vectors inside the sphere are tested.

The advantage of this method is that we never test vectors with a norm greater than the given radius. Every tested vector requires the computation of its norm, which entails n multiplications and $n-1$ additions. The increase in the number of operations needed to update the bounds (6) is largely compensated for by the enormous reduction in the number of vectors tested especially when the dimension increases.

In order to be sure to always find a lattice point inside the sphere we must select $\sqrt{C}$ equal to the covering radius of the lattice. Otherwise, we do bounded distance decoding and the decoder can signal an erasure whenever no point is found inside the sphere. A judicious choice of C can greatly speed up the decoder. In practice, the choice of C can be adjusted according to the noise variance N_0 so that the probability of a decoding failure is negligible. If a decoding failure is detected, the operation can either be repeated with a greater radius or an erasure can be declared.

The kernel of the universal decoder (the enumeration of lattice points inside a sphere of radius $\sqrt{C}$) requires the greatest number of operations. The complexity is obviously independent from the constellation size, i.e., the number of operations does not depend on the spectral efficiency of the signal constellation.

The complexity presented in [7] shows that if d^{-1} is a lower bound for the eigenvalues of the Gram matrix G, then the number of arithmetical operations is

$$O\left(n^2 \times \left(1 + \frac{n-1}{4dC}\right)^{4dC}\right). \tag{7}$$

For a fixed radius and a given lattice (which fixes d), the complexity of the decoding algorithm is polynomial. We would like to notice that this does not mean that the general lattice decoding problem is not NP-hard. In fact, it is possible to construct a sequence of lattices of increasing dimension with an increasing value of the exponent d.

When we deal with a lattice constellation, we must consider the edge effects. During the search in the sphere we discard the points which do not belong to the lattice code; if no code vector is found we declare an erasure. The complexity of this additional test depends on the shape of the constellation. For cubic-shaped constellations it only entails checking that the vector components lay within a given range. For a spherically shaped signal set it is sufficient to compute the length of the code vector found in the search sphere in order to check if it is within the outermost shell of the constellation.

III. The Sphere Decoder with Fading

For ML decoding with perfect CSI at the receiver, the problem is to minimize metric (1). Let M be the generator matrix of the lattice Λ and let us consider the lattice Λ_c with generator matrix

$$M_c = M \,\mathrm{diag}\,(\alpha_1, \ldots, \alpha_n)$$

We can imagine this new lattice Λ_c in a space where each component has been compressed or enlarged by a factor α_i. A point of Λ_c can be written as $\boldsymbol{x}^{(c)} = (x_1^{(c)}, \cdots x_n^{(c)}) = (\alpha_1 x_1, \cdots \alpha_n x_n)$. The metric to minimize is then

$$m(\boldsymbol{x} \mid \boldsymbol{r}, \boldsymbol{\alpha}) = \sum_{i=1}^{n} \left| r_i - x_i^{(c)} \right|^2.$$

This means that we can simply apply the lattice decoding algorithm to the lattice Λ_c, when the received point is $\boldsymbol{r}$. The decoded point $\hat{\boldsymbol{x}}^{(c)} \in \Lambda_c$ has the same integer components $(\hat{u}_1, \cdots, \hat{u}_n)$ as $\hat{\boldsymbol{x}} \in \Lambda$.

The additional complexity required by this decoding algorithm comes from the fact that for each received point we have a different

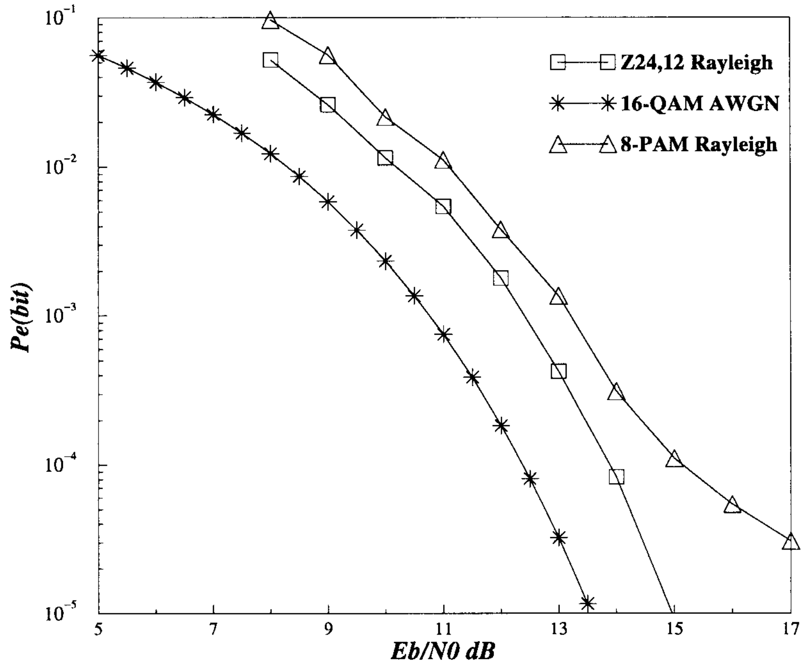

Fig. 3. Performance of the rotated lattice $\boldsymbol{Z}_{24,12}$.

compressed lattice Λ_c. So we need to compute a new Cholesky factorization of the Gram matrix for each Λ_c, which requires $O(n^3/3)$ operations. We also need $M_c^{-1} = \mathrm{diag}\,(1/\alpha_1, \cdots, 1/\alpha_n)M^{-1}$ to find the ρ_i's, but this only requires a vector-matrix multiplication since M^{-1} is precomputed. The complete flowchart of the algorithm is given in Fig. 2.

The choice of C in this case is more critical. In fact whenever we are in the presence of deep fades then many points fall inside the search sphere and the decoding can be very slow. This is also evident from the fact that the Gram matrix of Λ_c may have a very small eigenvalue which gives a large exponent d in (1). This problem may be partially overcome by adapting C according to the values of the fading coefficients α_i.

Fig. 3 shows the performance of the rotated lattice constellation $\boldsymbol{Z}_{24,12}$ on the Rayleigh channel with a spectral efficiency of 2 bits/dimension. This lattice is a rotated version of the cubic lattice in dimension 24 with a diversity order equal to 12 given in [5]. For all rotated cubic lattices in [5], we can set the search radius $C = 1$ and thus the enumeration complexity increases as $O(n^6)$ if we do not take into account the fading.

Fig. 3 compares the performance of the $\boldsymbol{Z}_{24,12}$ constellation to the 16-QAM on a Gaussian channel which has the same spectral efficiency. We observe that such a high modulation diversity can bring the bit error rate within 2 dB from the Gaussian channel's curve. To show the effectiveness of rotated constellations with respect to other TCM schemes especially designed for the fading chennel, we have also plotted the bit-error rate of the optimal 64-state TCM over an 8-PAM signal set. We recall that the asymptotic slope of the error curve reflects the diversity order of the coding scheme. Then, we observe that the diversity of the TCM scheme is much lower than the one of the rotated constellation.

IV. Conclusion

Decoding arbitrary signal constellations in a fading environment can be a very complex task. When the signal set has no structure it is only possible to perform an exhaustive search through all the constellation points. Some signal constellations, which can be efficiently decoded when used over the Gaussian channel, become hard to decode when used over the fading channel since their structure is destroyed. Fortunately, for lattice constellations this is not the case since the faded constellation still preserves a lattice structure and only a small additional complexity is required. The algorithm we presented was successfully run to simulate systems using lattice constellations of dimensions up to 32 which seem to be sufficent to approach the performance of the Gaussian channel when dealing with a Rayleigh fading one.

References

[1] J. H. Conway and N. J. Sloane, *Sphere Packings, Lattices and Groups*, 2nd ed. New York: Springer-Verlag, 1993.
[2] G. D. Forney Jr., "A bounded distance decoding algorithm for the Leech lattice with generalizations," *IEEE Trans. Inform. Theory*, vol. 35, pp. 906–909, July 1989.
[3] A. Gersho and R. M. Gray, *Vector Quantization and Signal Compression*. Boston, MA: Kluwer, 1992.
[4] J. Boutros, E. Viterbo, C. Rastello, and J. C. Belfiore, "Good lattice constellations for both Rayleigh fading and Gaussian channels," *IEEE Trans. Inform. Theory*, vol. 42, pp. 502–518, Mar. 1996.
[5] J. Boutros and E. Viterbo, "Rotated multidimensional QAM constellations for Rayleigh fading channels," in *Proc. 1996 IEEE Information Theory Workshop* (Haifa, Israel, June 9–13, 1996), p. 23.
[6] M. Pohst, "On the computation of lattice vectors of minimal length, successive minima and reduced basis with applications," *ACM SIGSAM Bull.*, vol. 15, pp. 37–44, 1981.
[7] U. Fincke and M. Pohst, "Improved methods for calculating vectors of short length in a lattice, including a complexity analysis," *Math. Comput.*, vol. 44, pp. 463–471, Apr. 1985.
[8] E. Viterbo and E. Biglieri, "A universal lattice decoder," in *14eme Colloq. GRETSI* (Juan-les-Pins, France, Sept. 1993), pp. 611–614.