Spatial Coupling for Distributed Storage and Diversity Applications

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***Abstract*—Low-density parity-check codes are considered for erasure channels, mainly Root-LDPC codes that include a special type of checknodes. Spatial coupling is applied on parity bits of a Root-LDPC ensemble designed for a channel with 4 block-erasure states and a maximal design rate of $3/4$ attaining double diversity. The advantage of spatial coupling is shown in the erasure plane as an improvement of a threshold boundary, under independent erasures. The spatial coupling maintains the double diversity because it connects parity bits only. The drawback of this partial coupling is a weak saturation of the threshold boundary towards the capacity boundary.**

I. Introduction

The current technology in both software and hardware, combined to mathematical tools driven by Coding Theory, Information Theory, and Communication Theory, is dramatically changing our society. New efficient techniques appeared recently for point-to-point communications and for information transmission and processing in networks, namely: network coding, coding for distributed storage, spatially-coupled graph codes, and lattices in high dimension for iterative decoding.
In nowadays information systems, Cloud Computing is a common concept where most of the resources are available online. Such a concept is of great interest because resources are evolving in an instantaneous and automatic manner. There exists in communication systems a sub-class of Cloud Computing referred to as Distributed Computing. Distributed Storage is yet another case of Distributed Computing where a file is stored in parts on multiple network nodes.
In the research work of interest in this paper, a network of nodes may represent a set of distributed data centers as assumed in standard distributed storage literature.

Error-control codes are one of the main tools for distributive computing/storage. Besides trivial repetition coding, a Reed-Solomon code or any Maximum Distance Separable (MDS) code is capable of reconstructing the source file [1]. The main objectives of distributed coding for storage are:
• Increase the reliability of data reconstruction. This constraint is directly linked to storage space utilization in data centers. The service sold by the cloud provider to an end-user has a level of reliability. One of the main challenges for cloud providers is to make a tradeoff between a level of reliability proposed to the client and the storage space dedicated to the data. Indeed the level of reliability depends on the amount of redundancy in the code structure while the storage space in a warehouse cluster has to be optimized.
• Minimize the required bandwidth per repair. Indeed, the general network traffic should not be hogged by the cross-rack traffic due to the recovery operations. Codes dealing with this constraint are called regenerating codes [2]. Upon failure of a node, the replacement node is able to download a portion of the data stored in neighboring nodes, not the entire data as in standard erasure codes, before recovering data prior to failure, Therefore, regenerating codes allow a drastic reduction in the amount of data downloaded for repair [3][4][5].
• Reduce the number of nodes to be contacted per repair, i.e. code locality has to be low. The concept of code locality was first defined by Gopalan et al. as the locality of information [6]. The code locality is given by the following definition in the standard context of independent failures (independent erasures).

Definition 1: Consider a $[n, k, d]_q$ linear code C over a finite field $\mathbb{F}_q$. A codeword in C is written as a vector of n symbols $(c_1, c_2, \ldots, c_n)$. Information locality, $Loc(C)$, of C is defined as:

$$Loc(C) = max_i \ Loc(c_i),$$

where a code symbol c_i has locality $Loc(c_i) = r$ if it can be recovered from accessing only r other code symbols, i.e. $c_i = \sum_{j=1}^{r} \lambda_j c_j$, where $\lambda_j \in F_q$.

The locality of a symbol is the number of symbols that need to be accessed in order to reconstruct this symbol. Fast reparation is ensured by a low locality of symbols. A new class of codes minimizing locality appeared in the literature. These codes are called Locally Repairable Codes and were introduced in [7] and developed in, e.g. [8],[9]. See [10] for a survey of coding in distributed storage.

For distributed storage applications, the choice of the decoding method is yet another critical challenge. In this paper, we assume that the symbols of a codeword are transmitted on a symbol erasure channel. Code design should allow for an iterative decoding process in order to avoid the high complexity of Maximum Likelihood (ML) decoding. When using an ML decoder, all nodes are to be contacted in order to repair the failed node resulting in a high decoding complexity and a high consumption of network bandwidth. A tradeoff is made between locality and the iterative decoding process. In fact, good locality can be obtained through the design of sparse parity-check matrices.

Both iterative decoding and locality aspects have motivated the interest in spatially-coupled low-density parity-check (LDPC) codes that can achieve the maximum-a-posteriori decoding threshold under iterative Belief Propagation (BP). Recently, spatial coupling was shown to saturate the BP threshold of LDPC ensembles on binary-input memoryless channels [13][14]. A new method for spatial coupling of LDPC ensembles, called forward layered coupling is described in [15]. The method is inspired from overlapped layered coding. Edges of local ensembles and those defining the spatial coupling are separately built. In [16], the forward layered spatial coupling was applied to Root-LDPC ensembles [11][12], in order to improve the coding gain and saturate the outage boundary. In the context of distributed computing, Root-LDPC ensembles present some interesting properties. Root-LDPC codes embed a pseudo-random structure (as in standard LDPC) and a deterministic structure (to combat the effect of a non-ergodic channel). The Root-LDPC construction guarantees that information bits attain maximal diversity under iterative decoding over non-ergodic channels (e.g. block-fading and block-erasure channels). An exact definition of diversity is given in the next section. Notice that Root-LDPC codes allow only information bits to attain the targeted diversity while parity bits are usually diversity-deficient. Hence, in this paper, we propose to apply spatially-coupled Root-LDPC to distributed storage applications.

Section II presents briefly the Root-LDPC code structure and the channel model. The reader is assumed to have a minimal background on coding for fading and erasure channels, mainly the notion of diversity [19]. The objectives to be addressed in this paper are presented in Section IV.

II. Channel coding model

Before describing our distributed system model, let us first recall the Root-LDPC construction and introduce the notion of diversity. Root-LDPC codes were designed to deal with non-ergodic (quasi-static) fading and attain a maximal diversity order allowed by the number of degrees of freedom in the block-fading channel and authorized by the Root-LDPC coding rate. Diversity is encountered in different communication systems according to two principal aspects:
- Diversity from error-control coding point of view. It consists in creating many replicas of the same information.
- Diversity from communication channel point of view. It corresponds to the number of available degrees of freedom while transmitting information.

The key idea of the Root-LDPC construction is to let information bit nodes receive belief-propagation (BP) messages including all available fading states.
Root-LDPC ensembles are multi-edge-type LDPC ensembles with specific properties [11]. For simplicity, let us consider a rate-$1/2$ LDPC code of length N, where N bits are transmitted on a block-fading channel with $L = 2$ fading states (denoted by colors in the next section). For this rate-$1/2$ ensemble used as an introductory example, four classes of bits are defined construction:
- On the first channel (ϵ_1): $N/4$ information bits referred to as $1i$ and $N/4$ parity bits referred to as $1p$.
- On the second channel fading (ϵ_2): $N/4$ information bits referred to as $2i$ and $N/4$ parity bits referred to as $2p$.

In the same way, for the rate-$1/2$ ensemble, there are two classes of checknodes named $1c$ and $2c$. Checknodes of type $1c$ are rootchecks for $1i$ and checknodes $2c$ are rootchecks for $2i$. The notion of a *rootcheck* is defined as follows.

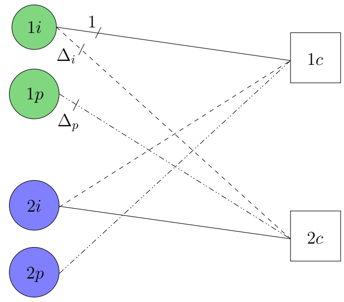

Figure 1. Compact Tanner graph representation for a rate-$1/2$ Root-LDPC ensemble. If the bits transmitted on one fading state are erased, erased information bits are recovered from other fading states.

Definition 2: Consider a checknode with a root bitnode transmitted on a fading channel ϵ_i. This checknode is a *rootcheck* with respect to its root if its leaf bitnodes undergo fadings ϵ_j, where $j \neq i$.

In order to guarantee full diversity for information bits (double diversity is the maximal diversity for rate $1/2$ with two channel states), information and parity bits are connected to checks $1c$ and $2c$ as shown by the compact Tanner graph of the rate-$1/2$ Root-LDPC ensemble in Figure 1. Edges between $1i$ and $1c$ are given by a graph matching (a permutation matrix in matrix representation). Each bitnode $1i$ has δ_i edges towards $2c$, the latter being a non-root checknode. Parity bits $1p$ are only connected to non-root checknodes via Δ_p edges. The root-LDPC structure is symmetric and an identical sub-graph is built for classes $2i$ and $2p$.

Under ML decoding, an upper bound on the diversity order d is given by the block-fading Singleton Bound [17] [18],

$$d \leq 1 + \lfloor L(1-R) \rfloor, \tag{1}$$

where R is the coding rate and L is the intrinsic channel diversity. For rate $R = 1/L$, Root-LDPC codes can attain a full-diversity order $d = L$ [12]. $R = 1/L$ is the maximal achievable coding rate for $d = L$ according to (1). The coding rate can exceed $1/L$ when $d < L$. As an example for distributed storage applications, rate-$3/4$ Root-LDPC codes yield $d = 2$ when $L = 4$. In this paper, fading is replaced by erasures which fits the coding model for distributive storage.

The different coding parameters are now restated in the context of distributed storage. Assuming a storage domain with a total of ℓ machines partitioned into $L = 4$ clusters. Each cluster has its own erasure probability ϵ_i corresponding to one state of the 4-state block-erasure channel. Here, block-fading channels are replaced by block-erasure channels. Indeed, a block erasure is a special case of block fading (the two extremal fadings 0 and ∞). Satisfying diversity d in distributed coding means that the code is capable of filling all erasures if $d-1$ clusters are erased. In our construction, the data stored in one cluster may be totally erased, diversity order $d = 2$ ensures that all these erasures will be filled. The channel model is depicted in Figure 2, where N bits are transmitted on four parallel BECs with erasure probabilities ϵ_i. Chunks of $N/4$ bits of the Root-LDPC are transmitted on each BEC(ϵ_i), $i = 1 \ldots 4$. Recall that for Root-LDPC codes, only the information bits are connected to rootchecks and have maximal diversity, leading to one cluster failures recovery.

However, the coding gain of such code is weak due the lower diversity of parity bits. In order to improve the coding gain and saturate the threshold (outage) boundary, we propose to apply forward-layered spatially-coupled Root-LDPC ensembles [16] to distributed storage. In the sequel, we shall use the coloring terminology to describe the four BEC channels with parameter ϵ_i: binary elements transmitted on BEC(ϵ_i) will be referred to as bitnodes of color i, $i = 1 \ldots 4$.

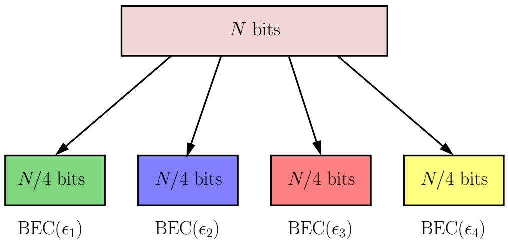

Figure 2. Parallel binary erasure channels where $N/4$ bits are transmitted on each channel BEC(ϵ_i), $i = 1 \ldots 4$. N is the total code length.

III. Uncoupled rate-$3/4$ Root-LDPC ensemble

For the channel model described at the end of the previous section, we split the first class of bits (green color) into information and parity bits respectively, i.e. $1i$ and $1p$. A similar splitting is made for all four colors leading to a total of eight classes of bitnodes as illustrated in Figure 3. Furthermore, four classes of checknodes are drawn on the right of the Tanner graph since the channel has $L = 4$ colors. Checknodes $1c$, $2c$, and $3c$ are rootchecks for class $1i$. Bitnodes $1p$ are connected to the non-root checknode class $4c$. The number of edges for information and parity bits towards the non-root checks is Δ_i and Δ_p. The root-LDPC structure is cyclic-symmetric. Similar edge connections are made for the three remaining colors. This ensemble shown in Figure 3 has rate $3/4$. For each color, there are $3N/16$ information bits and $N/16$ parity bits. There are $N/16$ checknodes of a given class, all checknodes have degree $d_c = 3 + 3\Delta_i + \Delta_p$. Each information bit has node degree $1 + \Delta_i$ and each parity bitnode has degree Δ_p. Then, the average bitnode degree d_b satisfies

$$d_b = \frac{3\Delta_i + \Delta_p + 3}{4} = \frac{d_c}{4}. \qquad (2)$$

The rootcheck structure guarantees double diversity in presence of block erasures. The proposed double-diversity maximal-rate code for four colors can also fill independent erasures. This capacity of handling both independent and burst erasures (per color) is studied now in the (ϵ_1, ϵ_2) plane. For simplicity, given the perfect cyclic-symmetry in the Root-LDPC ensemble, we assume that $\epsilon_2 = \epsilon_3 = \epsilon_4$ when drawing outage boundaries in the plane. The density evolution (DE) fixed points are given below without any constraint on the four channel parameters. DE fixed-points equations can be found by drawing the local neighborhood of each type of bitnodes.

Let us define the message densities for the uncoupled rate-$3/4$ root-LDPC ensemble as follows:

- f_1 is the density of message $1i \to 1c, 2c, 3c$.
- g_1 is the density of message $1i \to 4c$.
- q_1 is the density of message $1p \to 4c$.

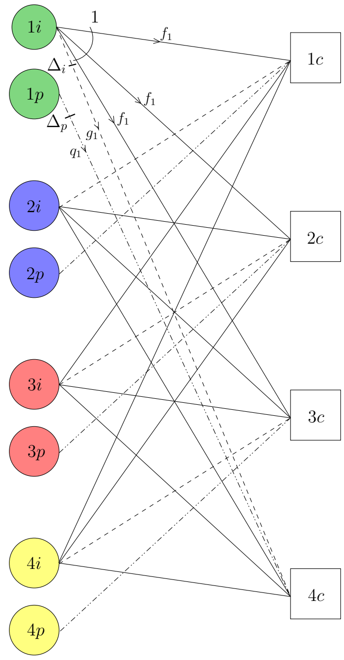

Figure 3. Compact Tanner graph representation for a rate-$3/4$ Root-LDPC ensemble for a channel with four colors.

Using the cyclic symmetry, messages f_i, g_i, and q_i are defined in a similar fashion for $i = 2, 3, 4$. Our LDPC ensemble corresponds to the following set of random graphs: edges from ($1i$,$1p$) towards $4c$ are constructed via a uniformly-selected socket permutation among the $(\Delta_i \frac{3N}{16} + \Delta_p \frac{N}{16})!$ permutations. The same construction is applied three more times for the three remaining colors. Hence, our rate-$3/4$ Root-LDPC ensemble is made out of four sets of interleavers. DE equations for the uncoupled ensemble are, for $i = 1 \ldots 4$:

$$f_i = \epsilon_i \left(1 - (1 - f_e g_i - g_e q_i)^{3\Delta_i + \Delta_p - 1} \prod_{j \neq i} (1 - f_j) \right)^{\Delta_i}, \qquad (3)$$

$$g_i = \epsilon_i \left(1 - (1 - f_e g_i - g_e q_i)^{3\Delta_i + \Delta_p - 1} \prod_{j \neq i} (1 - f_j) \right)^{\Delta_i - 1}$$
$$\cdot \frac{1}{3} \sum_{\substack{k=1 \\ k \neq i}}^{4} \left(1 - (1 - f_e g_k - g_e q_k)^{3\Delta_i + \Delta_p} \prod_{\substack{j \neq k \\ j \neq i}} (1 - f_j) \right), \qquad (4)$$

$$q_i = \epsilon_i \left(1 - (1 - f_e g_i - g_e q_i)^{3\Delta_i + \Delta_p - 1} \prod_{j \neq i} (1 - f_j) \right)^{\Delta_p - 1}, \qquad (5)$$

where the fractions f_e and g_e are defined as functions of the degrees between bitnodes and their non-root checknodes:

$$f_e = 1 - g_e = \frac{3\Delta_i}{3\Delta_i + \Delta_p}. \qquad (6)$$

The (ϵ_1, ϵ_2) plane, restricted to $[0,1]^2$, is partitioned into two regions, the outage region $\mathcal{R}$ and the non-outage region $\overline{\mathcal{R}}$. The non-outage region is

$$\overline{\mathcal{R}} = \left\{(\epsilon_1, \epsilon_2) \in [0,1]^2 \;:\; \mathbf{0} \text{ is the unique fixed point}\right\}. \quad (7)$$

The boundary between $\mathcal{R}$ and $\overline{\mathcal{R}}$ will be called the *outage boundary* and denoted by $\mathcal{B}_o$. It can be compared to the capacity boundary found by writing that the capacity of the four BEC channels is equal to the coding rate, i.e. the capacity boundary is given by the line

$$\epsilon_1 + 3\epsilon_2 = 1. \quad (8)$$

The boundary $\mathcal{B}_o$ is shown in Figure 4 for the root-LDPC ensemble with Tanner graph drawn in Figure 3. Double diversity in presence of block erasures (per color) is observed since $\mathcal{B}_o$ starts at the point $\epsilon_1 = 1$ and $\epsilon_2 = 0$. The BP threshold when all ϵ_i are equal is found on the so-called ergodic line ($\epsilon_1 = \epsilon_2$).

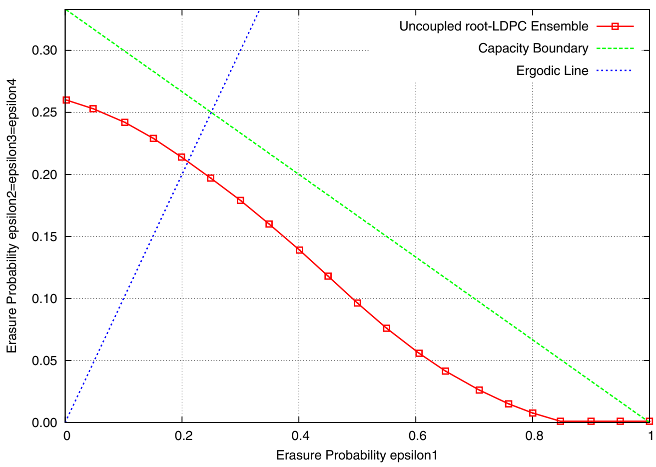

Figure 4. Outage boundary for rate-3/4 double-diversity root-LDPC. Graph parameters are $\Delta_i = 2$ and $\Delta_p = 3$.

In the following section, we propose a partial spatial coupling scheme for improving the boundary performance of the root-LDPC ensemble.

IV. Spatially Coupled Root-LDPC for Distributed Storage Applications

Spatially-coupled Root-LDPC ensembles were introduced in [16], in the case of two-state non-ergodic channels ($L = 2$) with a coding rate $1/2$. In this section, we build spatially-coupled Root-LDPC for four colors or equvalently four states ($L = 4$), where each state corresponds to a storage cluster.

The uncoupled ensemble of the previous section is copied L_c times, each copy being placed on a spatial position ℓ, where $\ell = 1 \ldots L_c$. In practice, the chain length L_c is taken to be large enough, e.g. $L_c = 100$. For the coupling scheme shown in Figure 5, the coupling window size is $w = 1$. Parity bitnodes at spatial position ℓ are coupled with non-root checknodes in spatial position $\ell + 1$. The chain is terminated at the right end with extra checknodes. For a general w, parity bitnodes at spatial position ℓ can be coupled with non-root checknodes in spatial positions $\ell+1, \ell+2, ..., \ell+w$. The chain is terminated by adding checknodes at w spatial positions at the right end. The number of edges per bitnode involved in coupling parity bits to the future w checknodes is Δ_c.

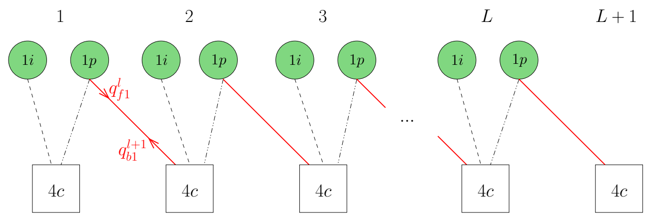

Figure 5. Spatial coupling structure for the root-LDPC ensemble with a coupling window size $w = 1$. Only parity bits are coupled to checknodes in the next spatial position.

For spatial coupling with $w = 1$, the following messages are defined at spatial position ℓ:

- f_i^ℓ, g_i^ℓ, and q_i^ℓ are the same messages as in (3)-(5), used to represent evolution in the local ensemble at position ℓ, for the four colors $i = 1 \ldots 4$.
- Messages f and g are not involved in spatial coupling, they are produced by information bits. For coupling parity bits, we keep the q notation. Message q_f stands for a forward message and q_b stands for a backward message. We define q_{fi}^ℓ as the left-to-right messages sent from bitnodes at position ℓ to checknodes at position $\ell + 1$. Similarly, q_{bi}^ℓ are backward messages from checknodes towards bitnodes in the previous spatial position, for $i = 1 \ldots 4$.

Then, DE equations for the spatially coupled ($w = 1$) rate-3/4 Root-LDPC ensemble are

$$f_i^\ell = \epsilon_i \left(1 - (1 - f_e g_i^\ell - g_e q_i^\ell)^{3\Delta_i+\Delta_p-1}\Pi_1\right)^{\Delta_i}, \quad (9)$$

$$g_i^\ell = \epsilon_i \left(1 - (1 - f_e g_i^\ell - g_e q_i^\ell)^{3\Delta_i+\Delta_p-1}\Pi_1\right)^{\Delta_i-1} \cdot \frac{1}{3} \sum_{\substack{k=1 \\ k\neq i}}^{4} \left(1 - (1 - f_e g_k^\ell - g_e q_k^\ell)^{3\Delta_i+\Delta_p}\Pi_3\right), \quad (10)$$

$$q_i^\ell = \epsilon_i \left(1 - (1 - f_e g_i^\ell - g_e q_i^\ell)^{3\Delta_i+\Delta_p-1}\Pi_1\right)^{\Delta_p-1} q_{bi}^{\ell+1\,\Delta_c}, \quad (11)$$

$$q_{fi}^\ell = \epsilon_i \left(1 - (1 - f_e g_i^\ell - g_e q_i^\ell)^{3\Delta_i+\Delta_p-1}\Pi_1\right)^{\Delta_p} q_{bi}^{\ell+1\,\Delta_c-1}, \quad (12)$$

$$q_{bi}^\ell = 1 - (1 - f_e g_i^\ell - g_e q_i^\ell)^{3\Delta_i+\Delta_p}\Pi_2, \quad (13)$$

where

$$\Pi_1 = \prod_{j\neq i} (1 - f_j^\ell)(1 - q_{fi}^{\ell-1})^{\Delta_c},$$

$$\Pi_2 = \prod_{j\neq i} (1 - f_j^\ell)(1 - q_{fi}^{\ell-1})^{\Delta_c-1},$$

and

$$\Pi_3 = \prod_{\substack{j \neq k \\ j \neq i}} (1 - f_j^\ell)(1 - q_{fk}^{\ell-1})^{\Delta_c}.$$

Density evolution fixed points for $w > 1$ are given by equations similar to the five equations listed above, we shall not include them in this paper. The outage boundary for the coupled Root-LDPC is shown in Figure 6 for a window size $w = 1$, 2, and 4. The new outage boundary approaches the capacity line $\epsilon_1 + 3\epsilon_2 = 1$. Nevertheless, we do not observe a boundary saturation as it is expected in spatially coupled LDPC codes. The reasons are detailed in the conclusions and are mainly due to the partial spatial coupling of parity bits only.

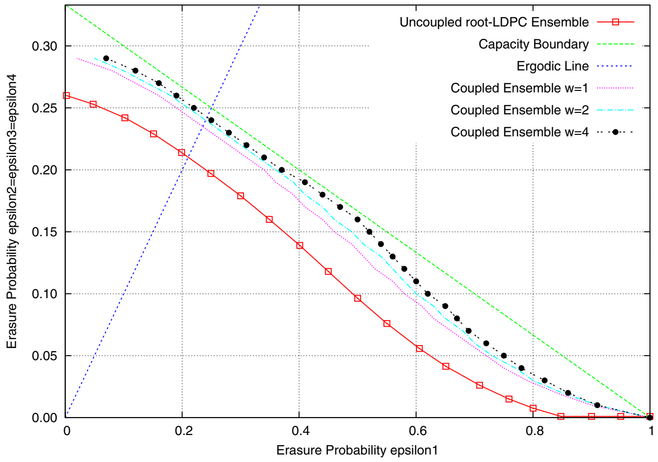

Figure 6. Boundary performance for rate-3/4 double-diversity root-LDPC with spatial coupling of window size $w = 1, 2, 4$. The graph parameters are $\Delta_c = w$, $\Delta_i = 2$, and $\Delta_p = 3, 5, 10$.

V. Conclusions

From Definition 1 and the structure of the double-diversity Root-LDPC codes, it is obvious that information bits have locality $d_c - 1 = 2 + 3\Delta_i + \Delta_p + \Delta_c$, i.e. the degree of checknodes minus one edge. In fact, this is true for any LDPC code when considering independent erasures. In the special case where erasures occur per color (block erasures) the locality of a random LDPC ensemble is $O(\log n)$ whereas that of a Root-LDPC is maintained at $d_c - 1$ thanks to root checknodes of order 1.

Usually, we would like to maintain all information bits at root order 1. This forces the construction to couple parity bits only. We also saw in the previous section that the outage boundary is not completely saturated towards the capacity line. Root order 1 for all information bits is too constraining for rate $3/4$ and 4 colors. A potential solution to saturate the boundary of coupled Root-LDPC (as a perspective for future work) is to introduce a fraction of order 2 information bits. This should allow the enhancement of spatial coupling by engaging both information and parity bits.

Acknowledgment

The work of Joseph J. Boutros and Volkan Dedeoglu was supported by QNRF, a member of Qatar Foundation, under NPRP project 5-603-2-243 at Texas A&M University.

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