Analysis of coding on non-ergodic block-fading channels

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Abstract

We study coding for the non-ergodic block-fading channel. In particular, we analyze the error probability of full-diversity binary codes, and we elaborate on how to approach the outage probability limit. In so doing, we introduce the concept outage boundary region, which is a graphical way to illustrate failures in the decoding process. We show that outage achieving codes have a frame error probability which is independent of the block length. Conversely, we show that codes that do not approach the outage probability have an error probability that grows logarithmically with the block length.

1 Channel model and notation

The block-fading channel is a simplified channel model that characterizes delay-constrained communication over slowly-varying fading channels [8]. Particular instances of the block-fading channel are orthogonal-frequency multiplexing modulation (OFDM) and frequency-hopping systems.

We consider a block-fading channel with n_c fading blocks, whose discrete-time channel output at time i is given by

$$y_i = h_i\, x_i + z_i \qquad i = 1 \ldots N_f \tag{1}$$

where N_f denotes the frame length, $x_i \in \{-1, +1\}$ is the i-th BPSK modulated symbol, $z_i \sim \mathcal{N}(0, \sigma^2)$ are the i.i.d. Gaussian noise samples, $\sigma^2 = N_0/2$, and h_i is a real fading coefficient that belongs to the set

$$\aleph = \{\alpha_1, \alpha_2, \ldots, \alpha_{n_c}\}. \tag{2}$$

The set $\aleph$ contains the possible fading coefficients (assumed i.i.d. Rayleigh distributed), it is fixed for the whole duration of the codeword, and changes independently from codeword to codeword. With a slight abuse of notation, we will also denote $\aleph$ as the vector of fading coefficients. We further assume that the fading coefficients are known to the receiver and not known to the transmitter. A simple illustration for the block-fading model is given in Figure 1. Strictly speaking, the capacity of the block-fading channel is zero as there is an irreducible probability that the decoder makes an error. In the limit of large block-length, this probability is the *information outage probability* defined as [8]

$$P_{out} = \Pr\{\mathcal{I}_\aleph \leq R\} \tag{3}$$

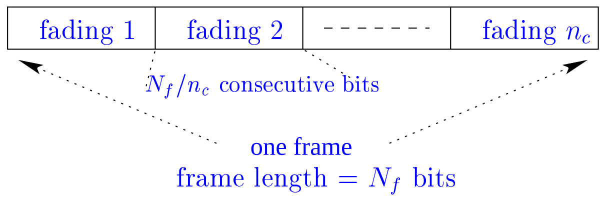

Figure 1: Block fading channel model for a binary (N_f, K_f) code (one frame).

where $\mathcal{I}_{\aleph}$ denotes the *instantaneous* mutual information between the input and output of the channel for a particular channel realization $\aleph$, and R is the transmission rate in bits per channel use. In [5] it is shown that for BPSK inputs, the asymptotic slope d of P_{out} with the signal-to-noise ratio in a log-log scale (commonly referred to as diversity) yields the optimal code design tradeoff, which is given by (Singleton bound)

$$d = 1 + \lfloor n_c(1 - R) \rfloor. \tag{4}$$

Therefore, in order to achieve reliable communication, we will design codes that achieve this optimal tradeoff. We consider linear binary block codes $C = (N_f, K_f)_2$ of length N_f, dimension K_f, and rate $R = K_f/N_f$. A codeword of C will also be called a *frame*. Two constructions are considered in this paper. Construction 1 assumes that C is the direct sum of N small block codes $C_0(n, k)_2$, where $N_f = N \times n$ and $K_f = N \times k$. This construction also includes the direct sum of N small Euclidean codes on a real channel, such as a space-time block codes on a multiple antenna channel. Construction 2 corresponds to a unique codeword per frame, such as using a parallel turbo code of interleaver size K_f and block length N_f. In this case, $N_f = n_c N$. The codes studied in this paper are *full diversity*, i.e., $R = 1/n_c$. We now introduce the notion of *channel multiplexer*, which is a particular type of interleaver that maps the output of the encoder to the n_c different channel coefficients.

Definition 1 *A multiplexer is a bijection from the integer set* $\{1 \ldots N_f\}$ *to the set* $\{1 \ldots n_c\}^N$*, where the superscript denotes Cartesian product.*

For a given code $C(N_f, K_f)$ transmitted on a non-ergodic channel with n_c states, the total number of multiplexers is $(N_f)!/(N!)^{n_c}$. A multiplexer is said to be 'regular' if it is has a periodical pattern. The number of regular multiplexers of period n reduces to $(n!)/((n/n_c)!)^{n_c}$. As an example, consider the $(8, 4, 4)$ linear binary code. For $n_c = 2$ channel states, a regular multiplexer is defined by the fading vector $(\alpha_1, \alpha_1, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_2, \alpha_2)$ or equivalently (11112222) applied on all $(8, 4, 4)$ codewords inside a frame. Example of multiplexers for parallel turbo codes are given in Figure 2

h-diagonal Multiplexer						
$\mathbf{s_1}$	1	2	1	2	1	2
$\mathbf{s_2}$	2	1	2	1	2	1
$\mathbf{s_3}$	3	3	3	3	3	3

h-π-diagonal Multiplexer						
$\mathbf{s_1}$	1	2	1	2	1	2
$\mathbf{s_2}$	2	X	2	X	2	X
$\mathbf{s_3}$	$\pi($ X	1	X	1	X	1 $)$

Figure 2: Multiplexers for near-outage performance from [1]. (Left) h-diagonal multiplexer for a rate 1/3 turbo code ($n_c = 3$). (Right), h-π-diagonal multiplexer for a rate 1/2 turbo code.

In this paper we study the error probability of binary codes in the block-fading channel. One of the objectives of this paper is to study the variation of the frame error probability with respect to the frame length, i.e. $P_{ef} = P_{ef}(N_f) = P_{ef}(N)$, under a fixed channel multiplexing. In particular, the outage formulation implies that outage-approaching codes not only should meet the Singleton bound, but also should perform close to the outage probability for large block-length. This implies that for a given channel realization, outage-approaching codes exhibit a threshold phenomenon, namely, the error probability goes to zero for $\sigma^2 < \sigma^2_{th}$, for some threshold noise variance σ^2_{th} (which depends on the fading realization) [5, 4]. In particular, we show that for classical short block codes or convolutional codes for which P_{ef} scales linearly with N in the ergodic channel, P_{ef} scales logarithmically with N in the non-ergodic case. In order to illustrate this effect, consider the simple case $n_c = 2$ and $C_0 = (8, 4, 4)$ drawn in Figure 3. It is clear that P_{ef} does not scale linearly with N (for a fixed E_b/N_0). This behavior has been noticed in [1] for convolutional codes. Obviously, such codes cannot approach the outage probability.

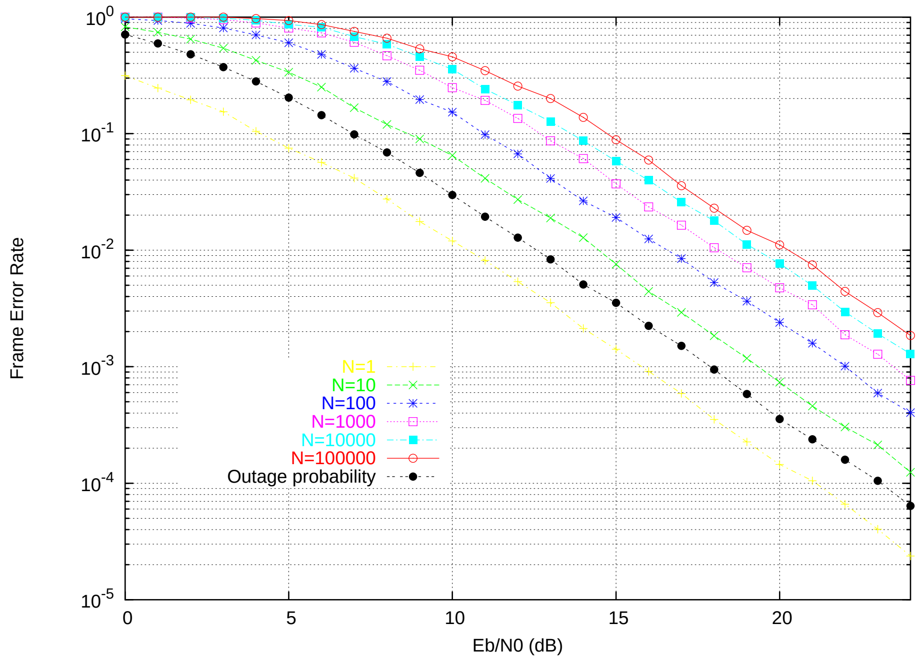

Figure 3: Frame error rate versus signal-to-noise ratio for $n_c = 2$ and $C_0 = (8, 4, 4)$. Gaussian channel with BPSK modulation, maximum-likelihood decoding.

2 Outage boundaries in the fading space

In this section, we introduce a new tool for analyzing the behavior of error-correcting codes on non-ergodic fading channels. The key idea is to study an outage representation in the Euclidean space defined by the n_c fading coefficients. The squared distance in the fading space is equivalent to an energy factor. A point $(\alpha_1, \ldots, \alpha_{nc})$ in that space is a channel instance. The line $\alpha_1 = \alpha_2 = \ldots = \alpha_{nc}$ is called the 'ergodic line' since it corresponds to the ergodic channel with a shift in signal-to-noise ratio. For a given code, the space is partitioned into two regions, an outage region D_o around the origin and a non-outage region for moderate and large fading values. The frontier separating these two regions is called the *outage boundary*, denoted by B_o. Let us first determine two important outage boundaries from the information theoretical properties of the channel.

Proposition 1 *The outage boundary $B_o(gauss)$ achievable by a real Gaussian channel with unconstrained input is defined by the set of fading points satisfying*

$$\prod_{i=1}^{n_c} \left(1 + 2R\frac{E_b}{N_0}\alpha_i^2\right) = 2^{2n_cR}$$

Proposition 2 *The outage boundary $B_o(bpsk)$ achievable by a real Gaussian channel with binary phase shift-keying input is defined by the set of fading points satisfying*

$$E_{\{X_i\}}\left[\log_2\left(\prod_{i=1}^{n_c}\left(1+e^{-\frac{2\alpha_i X_i}{N_0}}\right)\right)\right] = n_c(1-R)$$

where X_i are iid real Gaussian random variables with zero mean and unit variance.

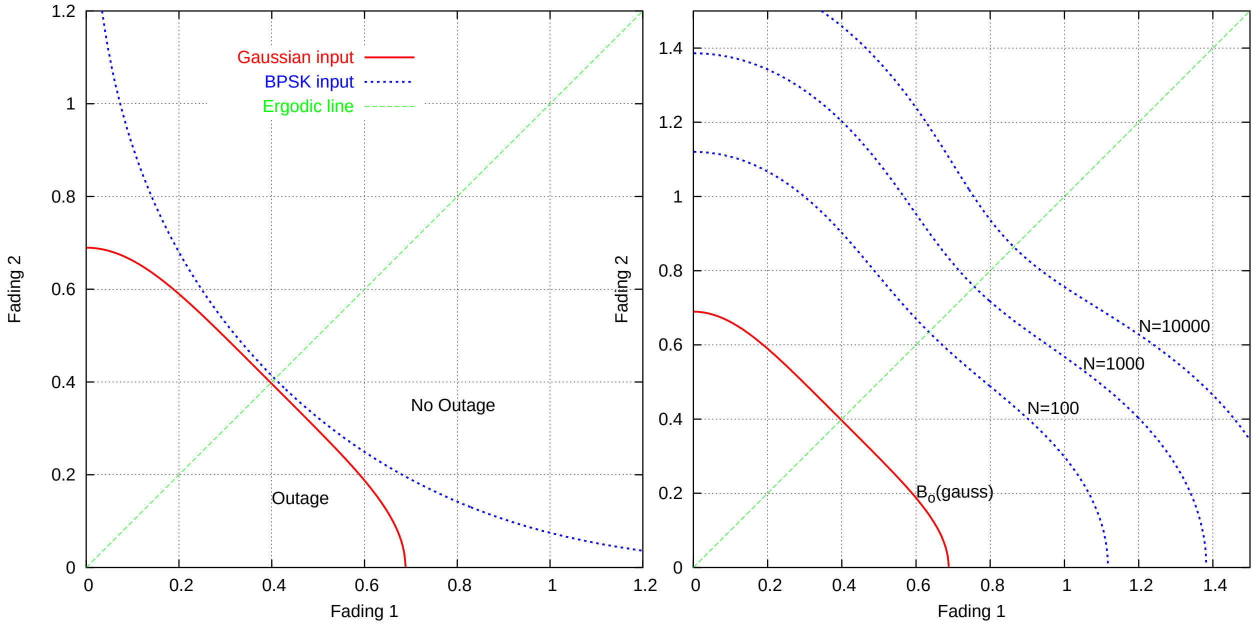

Figure 4: Outage boundaries in the fading plane, $n_c = 2$, $E_b/N_0 = 8$ dB. (Left), outage boundaries $B_o(gauss)$ and $B_o(bpsk)$ defined by propositions 1 & 2. (Right), outage boundary $B_o(C)$ for $C_0 = (8, 4, 4)$ and N=100, 1000, 10000 codewords per frame.

Let $P_{ew}(\aleph)$ denote the conditional word error probability of $C_0(n, k)$ for a given set of fading values. Then, the frame error probability of $C(N_f, K_f)$ (Construction 1) is

$$P_{ef} = \int p(\aleph)\ \left[1-(1-P_{ew}(\aleph))^N\ \right]\ d\aleph \tag{5}$$

We can now use the upperbound in [7] and write

$$P_{ef}\ \leq\ J = \int p(\aleph)\ \min\{1\ ,\ NP_{ew}(\aleph)\}\ d\aleph \tag{6}$$

Definition 2 *The outage boundary $B_o(C)$ achievable by a $C(N_f, K_f)$ code built by a direct sum of $C_0(n, k)$ and transmitted on a block fading channel is defined by the set of fading points satisfying*

$$P_{ew}(\aleph) = \frac{1}{N}$$

The outage boundaries $B_o(gauss)$, $B_o(bpsk)$ and $B_o((8,4,4))$ are plotted in Figure 4 in the bidimensional case.

In section 4 we study in more detail the outage boundary $B_o(C)$ for turbo codes under iterative decoding. We show that the outage boundary in the limit of large block length can be easily computed using density evolution.

3 Analysis of construction 1: Binary block codes

In this section we study the error probability of construction 1 using short block codes. For simplicity and without loss of generality, we assume that $n_c = 2$. The analysis will focus on the $(8,4,4)$ code, but the results are generalizable to other linear block codes and binary convolutional codes. The general behavior of P_{ef} with respect to N is similar for all non capacity-achieving error-correcting codes. Let $A(x) = \sum_i a_i x^i$ denote the weight enumerator polynomial of C_0 [6]. We define the *multiplexed weight enumerator* $A(x,y) = \sum_i \sum_j a_{ij} x^i\ y^j$, where a_{ij} is the number of C_0 codewords of Hamming weight $i+j$ for which the multiplexer assigns Hamming weight i to fading α_1 and Hamming weight j to fading α_2. Obviously $A(x,x) = A(x)$. A code C_0 associated to a given channel multiplexer has full diversity (see [1] for the definition of diversity and its relation to partial Hamming weights) if and only if $A(x,y) - 1$ is divisible by xy. The number of distinct multiplexed weight enumerators $A(x,y)$ depends on the algebraic structure of C_0. Some examples are given in Table 1, e.g. the (6,3,3) code obtained by shortening the (7,4,3) has 3 distinct $A(x,y)$ polynomials.

Block code $(\mathbf{n}, \mathbf{k}, \mathbf{d_H})_\mathbf{2}$	Total number of multiplexers	Number of multiplexing classes	Population per class	Diversity order
$(10,5,4)$	252	14	$6^2, 12^4, 24^4 \mid 24^4$	$1 \mid 2$
$(8,4,4)$	70	2	$14^1 \mid 56^1$	$1 \mid 2$
$(8,4,3)$	70	13	$1^2, 4^4, 8^3 \mid 4^1, 8^3$	$1 \mid 2$
$(6,3,3)$	20	3	$4^2 \mid 12^1$	$1 \mid 2$

Table 1: Some rate 1/2 block codes for $n_c = 2$ block fading channels.

The 70 multiplexers of the (8,4,4) extended Hamming codes are grouped in two classes only, with two distinct $A(x,y)$ polynomials, the first with diversity 1 is $A(x,y) = 1 + y^4 + 12x^2y^2 + x^4 + x^4y^4$ and the second with diversity 2 is $A(x,y) = 1 + 6x^2y^2 + 4x^3y + 4xy^3 + x^4y^4$.

Let us focus on the full-diversity $(8,4,4)$ case. From the expression of $A(x,y)$, the conditional word error probability is upper bounded as

$$
\begin{aligned}
P_{ew}(\aleph) \quad &\leq \quad 6Q\left(\sqrt{2R\tfrac{E_b}{N_0}(2\alpha_1^2 + 2\alpha_2^2)}\right) \ + \ 4Q\left(\sqrt{2R\tfrac{E_b}{N_0}(3\alpha_1^2 + \alpha_2^2)}\right) \\
&+ \quad 4Q\left(\sqrt{2R\tfrac{E_b}{N_0}(\alpha_1^2 + 3\alpha_2^2)}\right) \ + \ Q\left(\sqrt{2R\tfrac{E_b}{N_0}(4\alpha_1^2 + 4\alpha_2^2)}\right)
\end{aligned}
\tag{7}
$$

Let $D_o(C)$ represent the outage region of C with border $B_o(C)$. In (6), the upper bound J on P_{ef} can be decomposed into two integrals

$$
P_{ef} \ \leq \ J \ = \ \int_{\aleph \in D_o(C)} p(\aleph)\ d\aleph + \int_{\aleph \notin D_o(C)} N P_{ew}(\aleph)\ d\aleph \ \leq \ J_u \tag{8}
$$

where $J_u = J_u(1) + J_u(2)$ is obtained with the substitution of P_{ew} by its upper bound from (7). The first part $J_u(1)$ in the frame error probability is dominated by the channel fading (i.e. errors due to outage), the second part $J_u(2)$ is due to channel noise. Let us now determine how $J_u(1)$ and $J_u(2)$ scale with N. We introduce two parameters, R_1 and R_2 called *inner radius* and *outer radius* respectively. They are defined by (see Figure 5)

$$P_{ew}\left(\alpha_1 = \alpha_2 = \frac{R_1}{\sqrt{2}}\right) = P_{ew}(\alpha_1 = R_2, \alpha_2 = 0) = \frac{1}{N} \tag{9}$$

Using (7), the inner and outer radius are upper-bounded by

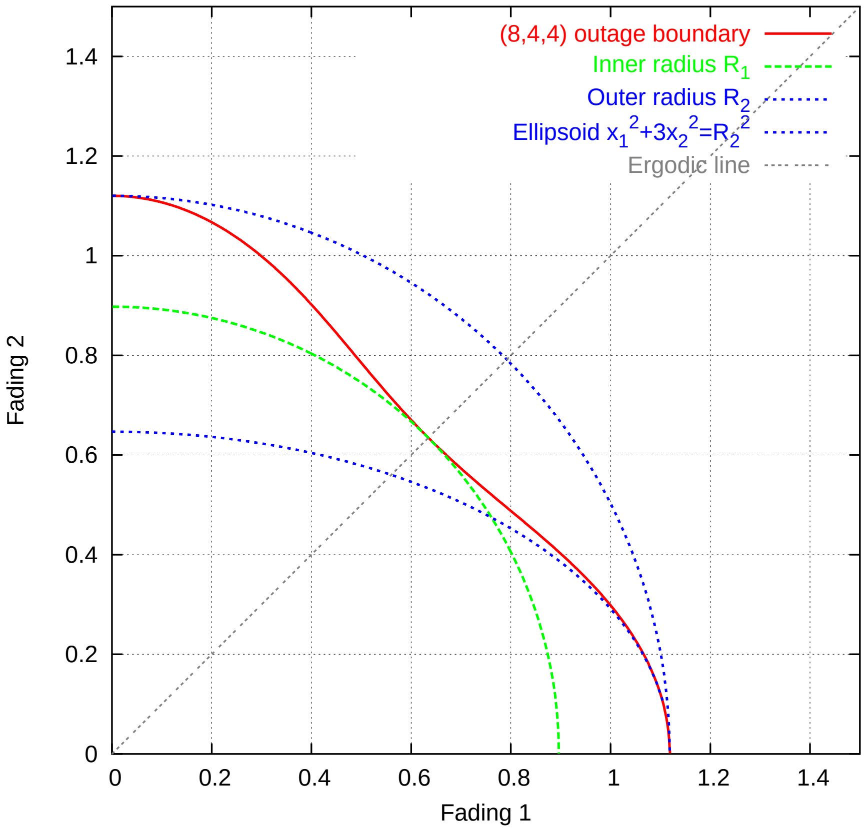

Figure 5: Outage boundary $B_o((8,4,4))$, the inner radius R_1, the outer radius R_2 and the ellipsoid $x_1^2 + 3x_2^2 = R_2^2$ defined by one monomial in $A(x, y)$.

$$R_1^2 \lessapprox \frac{\log(7N)}{2R\frac{E_b}{N_0}} \qquad R_2^2 \lessapprox \frac{\log(2N)}{R\frac{E_b}{N_0}} \tag{10}$$

Then, the integral $J_u(1)$ satisfies the following inequality

$$J_u(1) \leq \int_{\|\aleph\|^2 \leq R_2^2} p(\aleph)\, d\aleph = 1 - e^{-R_2^2}(1 + R_2^2) \lessapprox \frac{2\left[\log(2N)\right]^2}{\left(2R\frac{E_b}{N_0}\right)^2} \tag{11}$$

The second part $J_u(2)$ in (8) is an integral over $\aleph \notin D_o(C)$. This integral is the sum of 4 terms each corresponding to one occurence of the Gaussian tail function in (7). Upper-bounding the 4 terms in $J_u(2)$ is made as follows:

1. The integral in $J_u(2)$ associated to $(2\alpha_1^2 + 2\alpha_2^2)$ is performed in the region outside the circle of radius R_1, i.e. $\|\aleph\|^2 \geq R_1^2$. Denote this integral $I_{R_1}(1)$.
2. The integral in $J_u(2)$ associated to $(4\alpha_1^2 + 4\alpha_2^2)$ is also performed in the region $\|\aleph\|^2 \geq R_1^2$. Denote this integral $I_{R_1}(2)$.
3. The two integrals in $J_u(2)$ associated to $(\alpha_1^2 + 3\alpha_2^2)$ and $(3\alpha_1^2 + \alpha_2^2)$ are equal (by symmetry). They are performed over the fading region above the two associated ellipsoids, $\alpha_1^2 + 3\alpha_2^2 \geq R_2^2$ and $3\alpha_1^2 + \alpha_2^2 \geq R_2^2$. Denote this integral Γ.

The integral $I_{R_1}(1)$ and $I_{R_1}(2)$ are easily upper-bounded as

$$I_{R_1}(1) \lessapprox \frac{3}{7}\frac{[\log(7N)+1]}{\left(1+2R\frac{E_b}{N_0}\right)^2} \qquad I_{R_1}(2) \lessapprox \frac{[2\log(7N)+1]}{98N\left(1+4R\frac{E_b}{N_0}\right)^2} \propto \frac{\log(N)}{N} \tag{12}$$

The contribution of $I_{R_1}(2)$ to $J_u(2)$ is negligible. The derivation of the last term Γ is made via three consecutive splits leading to 8 integrals. The contribution of integrals in $O(1/N^2)$ is neglected leading to

$$\Gamma \lessapprox \frac{[2\log(2N)+3]}{\left(2R\frac{E_b}{N_0}\right)^2} \tag{13}$$

Finally, by adding $J_u(1)$ and $J_u(2)$, the upper bound on the frame error probability of a $C(N_f, K_f)$ constructed by the direct sum of N binary $(8,4,4)$ codes is

$$P_{ef} \lessapprox \frac{[2[\log(2N)]^2 + 3/7\log(7N) + 2\log(2N) + 24/7]}{\left(2R\frac{E_b}{N_0}\right)^2} \tag{14}$$

The ratio of the simulated error rate by the above upper bound is shown on Figure 6. The dominant term is always given by $J_u(1)$.

For other short block codes or convolutional codes, define ω_m as the minimum of all i and j in monomials $x^i y^j$ found in $A(x,y)-1$. Also, denote A_{w_m} the coefficient a_{ij} of the corresponding monomial. Then, the dominant term in (14) becomes

$$\frac{\left[\log(\frac{NA_{w_m}}{2})\right]^2}{\left(\omega_m R\frac{E_b}{N_0}\right)^2} \tag{15}$$

4 Analysis of Construction 2: DE for turbo codes

In this section we study the error probability of sufficiently long turbo-codes under iterative decoding using density evolution. We can rewrite (5) as

$$P_{ef} = \int p(\aleph) P_{ew}(\aleph)\; d\aleph \tag{16}$$

where now $P_{ew}(\aleph)$ is the frame error probability of the turbo-code for a given channel realization $\aleph$. It is well known that turbo-codes exhibit a threshold phenomenon in the ergodic channel. In the non-ergodic case we can write that

$$P_{ef} = \int_{\aleph \in D_o(C)} p(\aleph)\; d\aleph \tag{17}$$

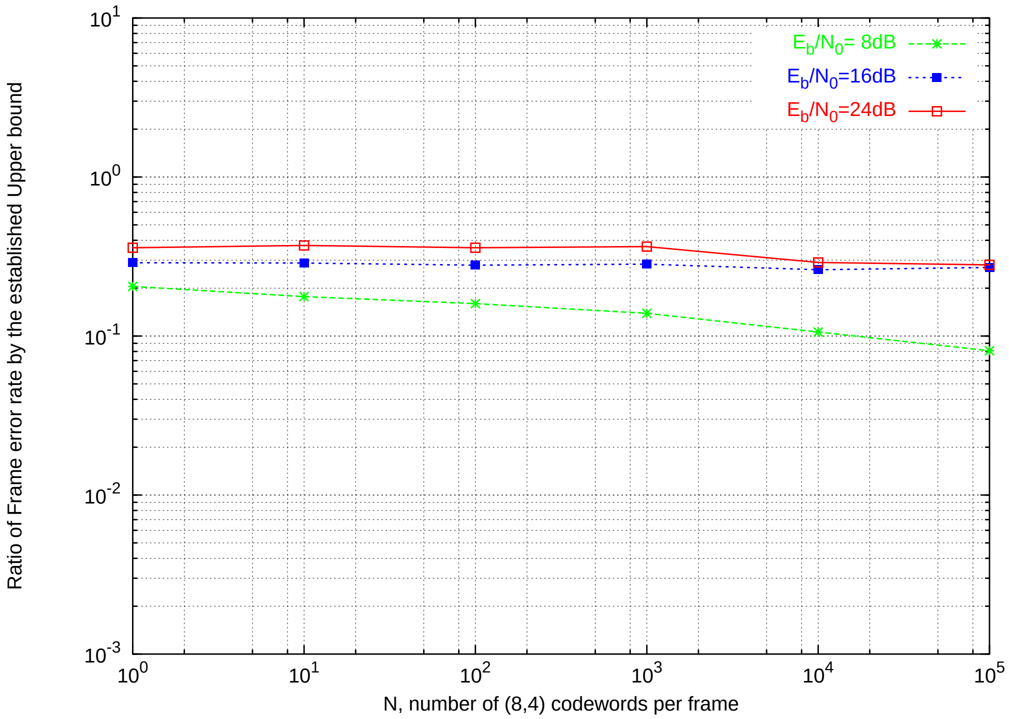

Figure 6: Ratio of the Monte Carlo simulated frame error rate by the upper bound established in (14), $C_0 = (8, 4, 4)$.

where

$$D_o(C) = \left\{ \aleph \in \mathbb{R}_+^{n_c} \mid \lim_{\ell \to \infty} \lim_{N \to \infty} P_{ew}^{\ell}(N) = 1 \right\} \tag{18}$$

denotes the outage region of the turbo code and ℓ denotes the number of iterations of the iterative decoder. In words, $D_o(C)$ is the fading region such that, for a fixed SNR the iterative decoder will not be able to decode. Thus, the error probability for large block length is given by the distribution of the decoding threshold as a function of the fading.

Brute force computation of this threshold distribution can be very complex, as for every fading realization we need a run of the density evolution algorithm. In order to reduce the complexity of the brute force density evolution algorithm we only run the algorithm when there is no outage and when $\alpha_{min} \frac{E_b}{N_0} > \frac{E_b}{N_0}\big|_{th}$, where $\alpha_{min} = \min \aleph$ and $\frac{E_b}{N_0}\big|_{th}$ is the iterative decoding threshold of the turbo-code in the AWGN channel. This preprocessing stage severely accelerates the overall algorithm and makes it practical.

It is also possible to efficiently compute the outage boundary $B_o(C)$. Figure 7 clearly illustrates the process. We take a grid of fading coefficients orthogonal to the BPSK outage curve and compute density evolution in all of them. If for a given fading the density evolution algorithm does not converge (thin dots), we move to the next point in the grid and run it again. The first point where the algorithm converges defines the boundary.

Figure 8 illustrates the density evolution method used to compute the error probability. As we observe, the h-π-diag channel multiplexer is about 0.8 dB from the outage probability limit. Note that this density evolution method for very large block length gives the same results as the finite-length simulations shown in [1]. Therefore, this turbo-code and multiplexer can approach the outage probability limit for *any* block length.

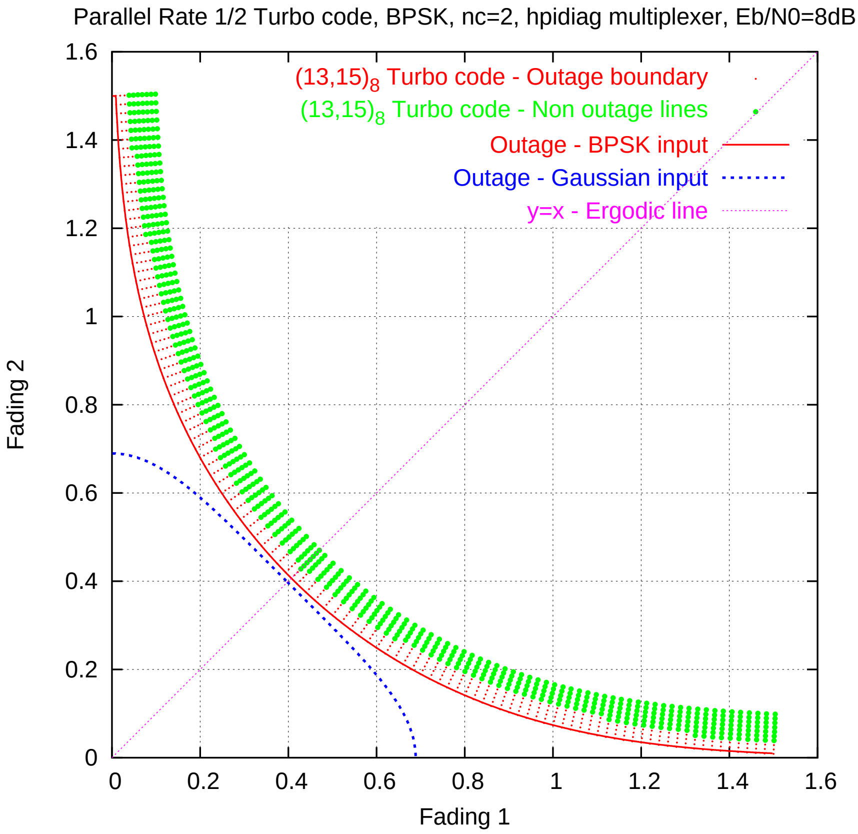

Figure 7: Boundary region $B_o(C)$ for the turbo-code of rate $R = 1/2$ with iterative decoding in the block-fading channel with $n_c = 2$. The thin dots denote the fading values where the density evolution does not converge. The thick dots denote the fading values where the algorithm converges.

5 Conclusions

We have studied the error probability of binary codes in the block-fading channel and we have shown that short block codes and convolutional codes cannot achieve the outage probability limit as their frame error rate grows as the block length increases. We have also shown that the growth rate is logarithmic. In so doing, we have introduced the outage boundary regions, which are a useful graphical tool to understand the source of errors in the decoding process. Furthermore, we show that turbo-codes with good channel multiplexers have frame error probability independent of the block-length and we have computed the asymptotic limit using density evolution.

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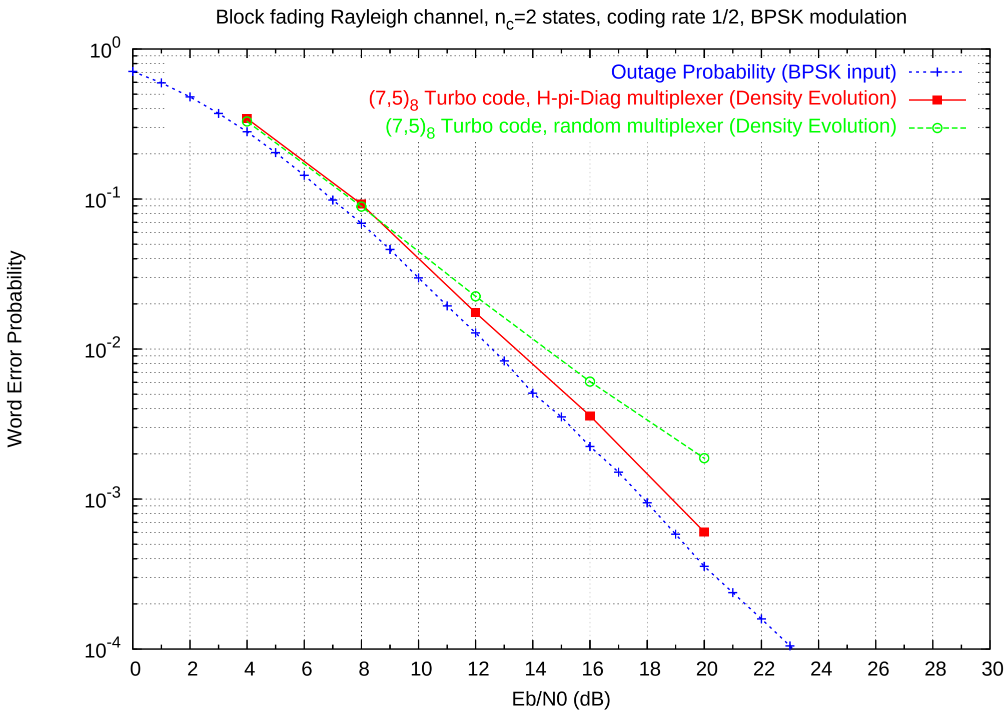

Figure 8: Error probability computed with the density evolution algorithm for $n_c = 2$ and the rate $R = 1/2$ turbo code with generators $(7, 5)_8$ with random and h-π-diag channel multiplexers.

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