Advanced Digital Communications and Coding

EXERCISES

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Exercice I - The Sphere Decoder, an example with A_2

Consider the hexagonal point lattice $A_2 \subset \mathbb{R}^2$. The studied lattice version is generated by the following generator matrix M written in row convention

$$M = \begin{pmatrix} 1 & 0 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \tag{1}$$

A lattice point $x = (x_1 \; x_2) \in A_2$ is defined by $x = zM$, where $z = (z_1 \; z_2) \in \mathbb{Z}^2$. Hexagonal lattice points and their Voronoi cells are illustrated in Figure 1. Let d_{Emin} denote the minimum Euclidean distance of A_2 generated by the above matrix M. The packing radius $\rho = d_{Emin}/2$ and the covering radius R are depicted in Figure 2.

1) Determine ρ and R.
2) What is the volume of a Voronoi cell ?
3) Check that the packing density Δ of A_2 is equal to 0.906, i.e. 90.6% of the bidimensional space is covered by the packing balls of A_2.
4) Let $G = MM^t = [g_{ij}]$ be the Gram matrix. We define the following quadratic form $Q(z)$ associated to the squared Euclidean norm

$$Q(z) = \|x\|^2 = xx^t = zGz^t = \sum_i \sum_j g_{ij} z_i z_j \tag{2}$$

Prove that $Q(z)$ can be written as $Q(z) = q_{11}Z_1^2 + q_{22}Z_2^2$, where $Z_1 = z_1 + q_{12}z_2$ and $Z_2 = z_2$. Find the values of q_{ij}. For higher dimensions $n \geq 2$, a Cholesky or a QR decomposition is applied in order to write $Q(z) = \sum_{i=1}^{n} q_{ii} Z_i^2$.

5) The enumeration of lattice points satisfying $\|x\|^2 \leq C$, where $C \in \mathbb{R}^+$ is a fixed squared radius, is equivalent to solving $Q(z) \leq C$. Let $S(0, \sqrt{C})$ denote the sphere with center 0 and radius $\sqrt{C}$. Prove that enumerating lattice points inside $S(0, \sqrt{C})$ is equivalent to solving the following recursive inequalities

$$
\begin{aligned}
-\sqrt{\tfrac{C}{q_{22}}} \;&\leq\; z_2 \;\leq\; +\sqrt{\tfrac{C}{q_{22}}} \\
-\sqrt{\tfrac{C-q_{22}z_2^2}{q_{11}}} - q_{12}z_2 \;&\leq\; z_1 \;\leq\; +\sqrt{\tfrac{C-q_{22}z_2^2}{q_{11}}} - q_{12}z_2
\end{aligned}
\qquad (3)
$$

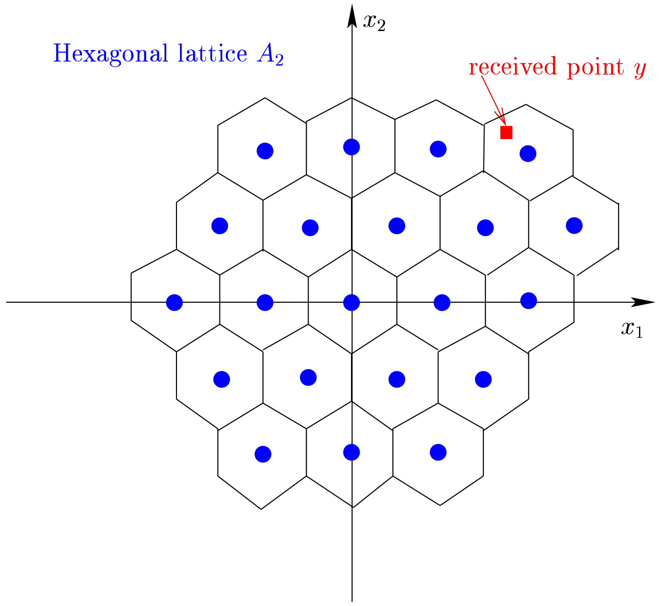

FIG. 1 – The hexagonal point lattice A_2 in the bidimensional plane.

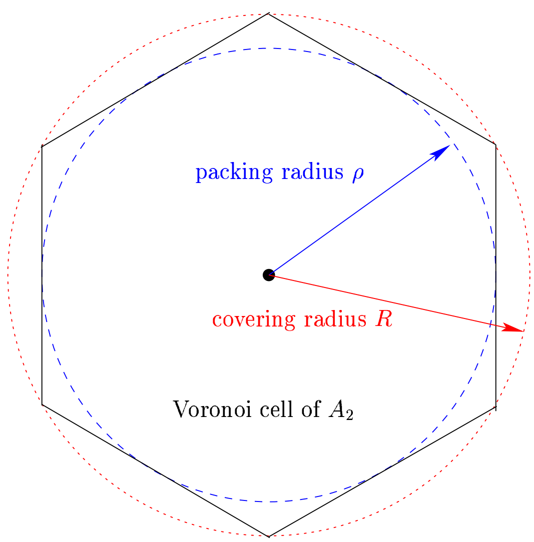

FIG. 2 – Representation of the packing radius ρ and the covering radius R.

6) Let $y = (y_1\ y_2) \in \mathbb{R}^2$ be a randomly chosen point, e.g. y may be the output of a Gaussian channel with $y = x + \eta$, where η is an additive white Gaussian noise and $x \in A_2$. Let ϵ be a

vanishing positive real number.
6.a) Prove that $S(y, R+\epsilon)$ contains at least one lattice point.
6.b) What is the maximum number of lattice points inside $S(y, R+\epsilon)$?

7) A sphere decoder is a decoder that finds the closest lattice point to y by enumerating all lattice points inside $S(y, R+\epsilon)$ and keeping the nearest one. Hence, the sphere decoder for A_2 takes $C = R^2 + \epsilon$ and solves $\|y - x\|^2 \leq C$ in order to get

$$x_{ML} = \arg \min_{x \in A_2} \|y - x\|^2$$

7.a) Prove that the sphere decoder is given by the inequalities of question 5 above, where z is replaced by $z - \xi$ and $\xi = yM^{-1}$.
7.b) Apply the sphere decoder in order to find the closest lattice point x_{ML} to the received noisy point given by $y = (1.75,\ 1.75)$.

Exercice II - The Gosset lattice E_8, complex construction

The point lattice E_8 yields the densest lattice packing in $\mathbb{R}^8$. It is the unique lattice in dimension 8 with an Hermite constant (fundamental gain) equal to 2, and a kissing number equal to 240. Recall that the fundamental gain of a real lattice $\Lambda \in \mathbb{R}^n$ of rank n is defined as

$$\gamma(\Lambda) = \frac{d^2_{Emin}(\Lambda)}{\sqrt[n/2]{vol(\Lambda)}} \tag{4}$$

where $vol(\Lambda) = |det(M)|$, M being a square $n \times n$ generator matrix of Λ. The kissing number τ is given by the number of lattice points on the first lattice shell, also equal to the number of neighbouring packing balls tangent to the one centered on the origin.

Let $\mathcal{G} = \mathbb{Z}[i] \sim \mathbb{Z}^2$ denote the ring of Gaussian integers. Let $\phi = 1+i \in \mathcal{G}$, where $i = \sqrt{-1}$. The ring $\mathcal{G}$ can be partitioned via two subgroups, i.e. the partition chain $\mathcal{G}/\phi\mathcal{G}/\phi^2\mathcal{G}$ is used. At depth 1, we have $\mathcal{G} = \phi\mathcal{G} + [\mathcal{G}/\phi\mathcal{G}]$. At depth 2, we have $\mathcal{G} = \phi^2\mathcal{G} + [\phi\mathcal{G}/\phi^2\mathcal{G}] + [\mathcal{G}/\phi\mathcal{G}]$.

1) Find the order of the quotient groups $[\mathcal{G}/\phi\mathcal{G}]$ and $[\phi\mathcal{G}/\phi^2\mathcal{G}]$. Tip : Determine the fundamental volumes of both lattices $\mathcal{G}$ and $\phi\mathcal{G}$. Find also $|\mathcal{G}^N/g\mathcal{G}^N|$, where $g \in \mathcal{G}$ and $N \in \mathbb{N}$.
2) Give the typical coset leaders of the partition $\mathcal{G}/\phi\mathcal{G}$, i.e. the typical elements of the quotient group $[\mathcal{G}/\phi\mathcal{G}]$. Use Figure 3 for illustration.
3) Give the typical coset leaders of the partition $\phi\mathcal{G}/\phi^2\mathcal{G}$, i.e. the typical elements of the quotient group $[\phi\mathcal{G}/\phi^2\mathcal{G}]$. Use Figure 4 for illustration.

4) Consider the complex lattice $\Lambda = (4,1,4) + \phi(4,3,2) + \phi^2\mathcal{G}^4$, where $(4,1,4)$ is the binary repetition code of length 4, and $(4,3,2)$ is the binary single parity-check code of length 4. In the formula of Λ, binary elements 0 and 1 should be embedded into the complex ring $\mathcal{G}$.
4.a) Consider the chain $\mathcal{G}^4/\Lambda/\phi^2\mathcal{G}^4$. Deduce the fundamental volume of Λ.
4.b) Determine the minimum Euclidean distance of Λ and hence its Hermite constant. At this point, you must find that $\Lambda = E_8$.
4.c) From the complex formula of E_8, check that $\tau = 240$ by enumerating all lattice points of shortest length.

5) An encoder for a finite size constellation carved from E_8 is shown in Figure 5. This encoder is based on the formula $E_8 = (4,1,4) + \phi(4,3,2) + \phi^2\mathcal{G}^4$ (B construction).
5.a) What is the number of bits per complex symbol in the coded 64-QAM ? What should be the uncoded QAM reference constellation ?
5.b) Compare the coded 64-QAM and the uncoded reference, and find again the 3dB gain over a Gaussian channel given in the Hermite constant.

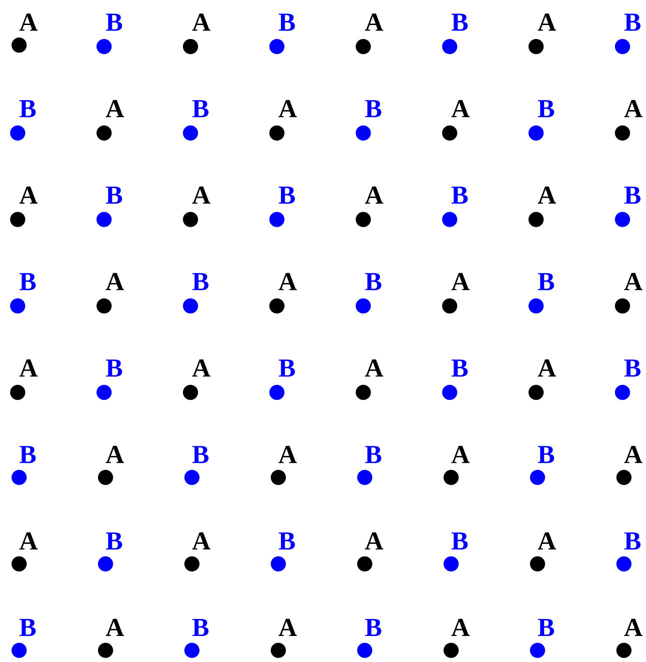

FIG. 3 – Depth-1 partitioning of a 64-QAM corresponding to $\mathcal{G} = \phi\mathcal{G} + [\mathcal{G}/\phi\mathcal{G}]$.

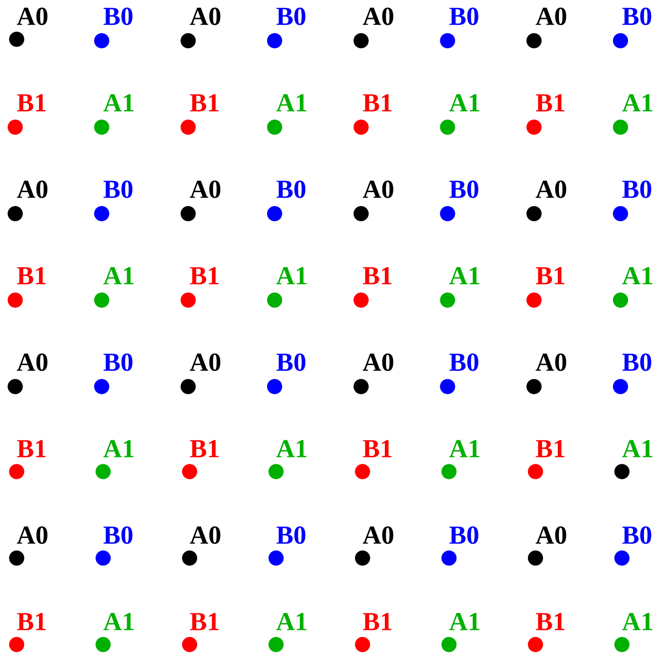

FIG. 4 – Depth-2 partitioning of a 64-QAM corresponding to $\mathcal{G} = \phi^2\mathcal{G} + [\phi\mathcal{G}/\phi^2\mathcal{G}] + [\mathcal{G}/\phi\mathcal{G}]$.

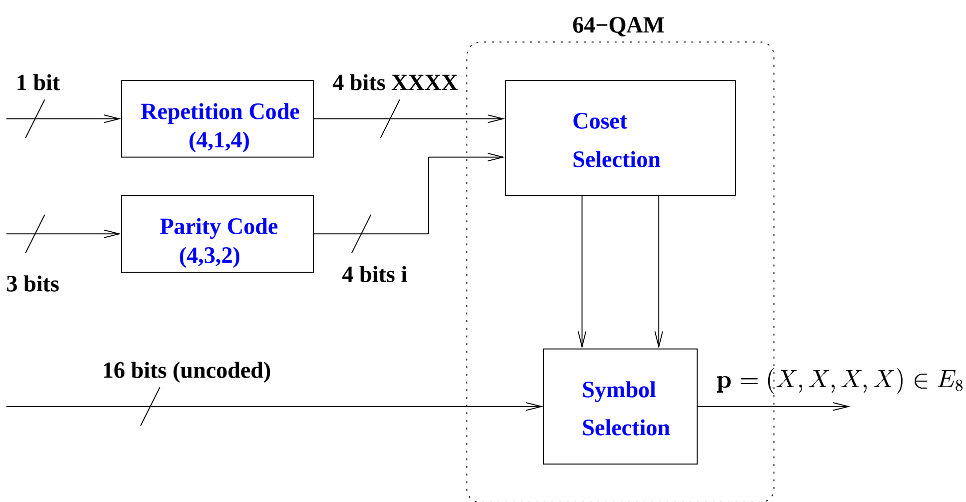

FIG. 5 – Encoder for an E_8 constellation. Information rate is 5 bits per complex dimension.