Full-diversity Product Codes for Block Erasure and Block Fading Channels

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Abstract

We show how to build full-diversity product codes under both iterative encoding and decoding over non-ergodic channels, in presence of block erasure and block fading. The concept of a rootcheck or a root subcode is introduced by generalizing the same principle recently invented for low-density parity-check codes. We also describe some channel related graphical properties of the new family of product codes, a family referred to as root product codes.

1 Introduction

Product codes are powerful compound codes with rich and elegant graphical and algebraic structures. Their error and erasure correcting capabilities in both bursty and non-bursty modes have been extensively studied in the two decades following their invention by Elias [8]. One of the simplest methods for combining two codes is to form their direct product, see [13], Chap. 18. Besides its nice algebraic properties [4][12][13], a product code has a graphical representation that can lead to even more powerful generalizations under both iterative encoding and decoding [20]. The interest in product codes has been propelled by the results obtained from iterative soft decoding that yields an excellent performance on classical ergodic Gaussian channels [16]. Several studies have been carried out on decoding product codes [1][9][15], analyzing their asymptotic and low error rate performance [7][17][18], unveiling more properties of their weight distribution [21], proposing design criteria and analyzing their erasure rate in the presence of ergodic i.i.d erasures [2][22], and describing the convergence of their iterative decoding [19], though not as much as the huge literature that exists on Turbo and LDPC codes.

In this paper, erasures and fadings encountered during transmission are not independent from one binary digit to another, they occur in blocks and are constant within a block, e.g. see [3], Chap. 4. The exact channel model will be given in section 3. Given a data transmission channel with n_c internal states, where n_c is referred to as the channel diversity order, an error-correcting code achieves a d-diversity order after decoding, e.g. on a BSC channel with transition probability p, the error rate at the decoder output would be written in the form $P_e \propto p^d$. The code is *full diversity* if $d = n_c$. In the absence of unit-rate linear precoding before transmission [10] (e.g. a unitary transformation), the block fading Singleton bound [11][14] implies that $d \leq \lfloor n_c(1 - R_c) \rfloor + 1$ for a rate R_c code. A code achieving the Singleton bound is referred to as *maximum distance separable* (MDS). An MDS code is not necessarily full diversity. However, a full-diversity code is necessarily MDS with coding rate $R_c \leq 1/n_c$.

Properties and methods for designing full-diversity product codes are studied in this paper. Our study is restricted to bi-dimensional binary product codes in the finite-length case. Some similarities exist between our product codes and array codes such as the dual of B-codes [23]. Both families are MDS, but array codes are not full-diversity and they are designed for channels with a relatively large diversity ($n_c = 5, 6$ or more) while our codes are meant for channels with limited diversity (n_c =2 or 3). Before introducing the *rootcheck* concept which is the main ingredient for the design of full-diversity product codes under iterative encoding and decoding, section 2 below summarizes the principal ideas and difficulties via the illustration of a simple 3×4 product code.

2 Problem Illustration

We illustrate the problem studied in this paper by setting block erasures on a simple product code. Consider the product code $C = [3, 2] \otimes [4, 3]$ of rate $R = 1/2$ and length 12, where $[n, n - 1]$ represents the linear binary single parity-check code of length n. Recall that a codeword of C can be written in a 3×4-matrix form, where each row is a codeword of $[4, 3]$ and each column is a codeword of $[3, 2]$. The code C is a $[12, 6, 4]$ code, it can be encoded via a 6×12 generator matrix and decoded via a 6×12 parity-check matrix. In this paper, we are only interested in encoding and decoding of a product code based on its row and column constituents.

The matrix structure of codewords belonging to C is shown in Figure 1. A box is associated to a binary element. Half of the boxes are colored in white and the other half in red. Assume that all red boxes are erased, i.e., all values of associated bits are lost, is the code C capable of finding them with the help of white boxes? Similarly, when white boxes are all erased, is the code capable of filling those erasures with the help of red boxes?
Four different colorings of the product code are given in Figure 1. In practice, they are equivalent to four different channel interleavers (also known as multiplexers) applied on code symbols. Consider the product code defined by Figure 1(a). If all red bits are erased, the decoding of the first two rows followed by the decoding of the 4 columns will fill the erasures. Unfortunately, if white bits are all erased, an infinite number of row-column decoding iterations will never retrieve the erased values. Hence, in the terminology of

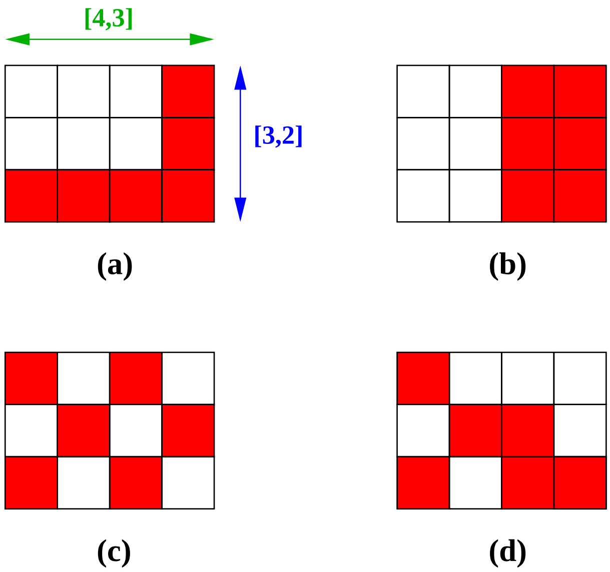

Figure 1: Four different channel interleaving of code symbols. In the context of non-ergodic channels, the 4 configurations define 4 different product codes.

communication theory, we would say that code 1(a) is not full diversity. The reader can check that 1(b) and 1(c) are not full-diversity codes.

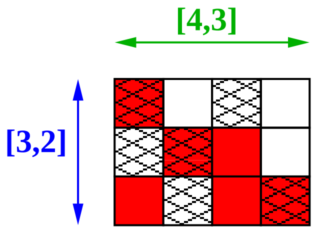

Figure 2: The full-diversity product code defined in Figure 1(d). Privileged bits are indicated by a pattern.

The product code defined in Figure 1(d) is full diversity. Indeed, if all red bits are erased, the decoder is capable of filling their values after 3 decoding iterations (row → column → row). Similarly, if all white bits are erased, they can be determined after 3 decoding iterations. The first bit (in red) on the first row is privileged because all other bits belonging to the same row are white. For the same reasons, the third bit on the first row (in white) is privileged because all other bits on the same column are red. A privileged bit, if erased, can be solved in one decoding iteration. Figure 2 depicts code 1(d) with privileged bits indicated by a pattern. Those bits are said to be connected to a rootcheck.

The standard location of information bits is the upper left 2×3 corner. The position of information bits can be moved to those connected to rootchecks. The new full-diversity code is referred to as a root product code. It has the following properties:

- It is full diversity on both block erasure and block fading channels.
- Full diversity is achieved after one decoding iteration. Indeed, erasures on information bits are filled in one decoding iteration (two iterations if we account for row and column decoding, since some bits are solved via their horizontal rootcheck, others are solved via their vertical rootcheck).
- It achieves the highest rate according to the block fading Singleton bound.

More efficient product codes should be considered, e.g., can we build a root product code $[16, 11]^{\otimes 2}$? Through an exhaustive search, the total number of red/white configurations is about 2^{250}! If all symmetries are taken into account, the number of configurations reduces to about $\approx 2^{160}$. The solution for the design of full-diversity product codes is found via the introduction of rootchecks in the code structure as described in section 4.

3 Channel Model and Notations

Linear binary coding for non-ergodic channels is considered. The channel state is assumed to be invariant for some time period, finite or infinite. Given the channel state α, an input $x = \pm 1$ and an output $y = \alpha x + \eta$, the channel transition probability is

$$p(y|x, \alpha) \propto \exp\left(-\frac{|y - \alpha x|^2}{2\sigma^2}\right),$$

where σ^2 is the variance of the additive white Gaussian noise η. Two cases are considered:

1. The non-ergodic Rayleigh fading channel where the fading coefficient α belongs to $\mathbb{R}^+$, with probability density function $2\alpha e^{-\alpha^2}$. We should emphasize that maximal diversity is still achieved in presence of other types of fading distribution, as in coding for MIMO channels [6] where a channel state is assigned a high order Nakagami distribution.
2. The block erasure channel where the fading coefficient α belongs to $\{0, +\infty\}$.

Within a codeword of length N bits, it is assumed that α takes n_c independent values. Also, the fading instances are supposed to be independent from one codeword to another. For simplicity, we consider the case $n_c = 2$ channel states per codeword, as illustrated in Figure 3. Code construction and analysis is generally straightforward for $n_c \geq 3$. Channel coding is made via a rate-R product code $C[N, K]$. The code C is built from a rate-r constituent $C_0[n, k]$, also referred to as a subcode of the product code. Thus, we have $C = C_0 \otimes C_0$, $N = n^2$, $K = k^2$, and $R = r^2$.

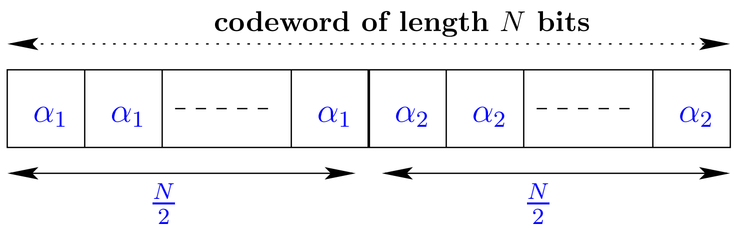

Figure 3: Data transmission channel with 2 states.

In the rest of this section, we briefly recall some fundamental properties of fading channels and their error-control coding. The channel diversity order is given by the number of independent channel states. The Tx/Rx diversity order, controlled by the system designer on the transmit and receive sides, is the number of independent replicas of the same information. At high signal-to-noise ratio $\gamma = \frac{1}{\sigma^2} \gg 1$, on a block fading channel of diversity order $d = n_c$, the word error rate behavior should be

$$P_e \propto \gamma^{-d}. \tag{1}$$

On a block erasure channel of diversity order $d = n_c$, for a given block erasure probability equal to ϵ, the word error rate behavior should be

$$P_e \ = \ \epsilon^d. \tag{2}$$

An error-correcting code whose error rate satisfies (1) and (2) after decoding is a full-diversity channel code. According to the block fading Singleton bound, the coding rate R of a full-diversity code is upper-bounded by:

$$R_{max} = \frac{1}{n_c} \geq R. \tag{3}$$

The two following propositions are essential in the design of coding for non-ergodic channels. Their proofs are simple and are left to the reader. For any code structure, under ML decoding, full diversity on block erasure channels is a necessary condition for full diversity on non-ergodic Rayleigh fading channels. This can be stated as follows

Proposition 1 *Consider a linear binary code $C[N, K]$. If C is full diversity on a block erasure channel then it is full diversity on a block fading channel under ML decoding.*

As described in the next section, the special root structure for any compound code achieves full diversity under iterative decoding. In this paper, we are concerned by product codes only. Other full-diversity codes such as Turbo, LDPC, and GLD/Tanner codes have been constructed by the authors, e.g. see [5] and references therein.

Proposition 2 *Consider a linear binary code $C[N, K]$ with a root structure (LDPC, Product, GLD, etc). Under iterative decoding, the code is full diversity on both block erasure and block Rayleigh fading channels.*

4 Root Checknodes for Linear Codes

A rootcheck is a special type of checknode suitable for designing codes on graphs matched to iterative decoding when transmitted over block fading and block erasure channels. In Tanner terminology [20], the constituent of a product code will also be called a *subcode*, or a *subcode node*. In our practical examples, we are mainly focusing on subcodes defined from the famous family of linear binary BCH codes [4][13].

Definition 3 *A rootcheck is a subcode node with all roots colored in white and all leaves colored in red. A similar definition is given after interchanging red and white.*

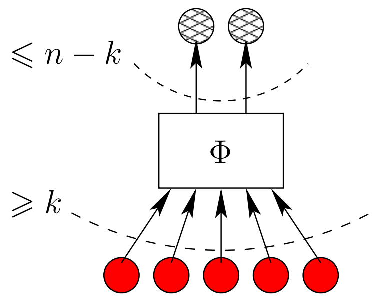

Figure 4: Structure of a rootcheck for a 2-state channel.

The definition of a rootcheck is illustrated by Figure 4. The version of the constituent C_0 defined by a parity-check matrix H_0 and used in a rootcheck must satisfy the following constraint:
The $n-k$ root vertices are assigned to $n-k$ independent columns of H_0. The simplest convention is to write the parity-check matrix in systematic form, $H_0 = [I_{n-k} \mid P_0]$, and assign the first $n-k$ columns to root bitnodes.

Proposition 4 ***Proposition 2 revisited at the subcode level (block erasure)***
A rootcheck $C_0[n, k, d]$ guarantees full diversity to all its roots under block erasures.

Proof : If root bits are erased then recompute their value from leaf bits using H_0. ∎

Proposition 5 ***Proposition 2 revisited at the subcode level (block fading)***
A rootcheck $C_0[n, k, d]$ guarantees full diversity to all its roots under block fading.

Proof : Local optimal probabilistic decoding is given by

$$APP(b) \propto \sum_{c \in C_0 | b} p(y|c), \tag{4}$$

where b is a root. The sub-optimal log-map decoder considers the dominant likelihoods for $b = 0$ and $b = 1$. Its log-ratio message is

$$\Lambda \; = \; \|y - \alpha \bar{x}\|^2 - \|y - \alpha x\|^2 \; = \; 2Y + \nu, \tag{5}$$

$$Y = \alpha_1^2 \sum_{i=1}^{n-k} (x_i - \bar{x}_i) \; + \; \alpha_2^2 \sum_{i=n-k+1}^{n} (x_i - \bar{x}_i) \; = \; \omega_1 \alpha_1^2 + \omega_2 \alpha_2^2. \tag{6}$$

At high SNR, we have $\omega_1 \geq 1$ and $\omega_2 \geq 1$. The exact values for ω_i depend on the weight distribution of C_0. The 4th order χ^2 distribution of Y guarantees the double diversity. ∎

5 Full-Diversity Product Codes

The primary role of a full-diversity code is to ensure the highest diversity order for its information symbols. Such a protection is not necessarily essential for parity-check symbols. Hence, the following definition applies the rootcheck concept to information digits only.

Definition 6 *A root product code is a product code where all information bits are covered by rootchecks, i.e., all information bits are rootbits belonging to a row or a column rootcheck.*

From propositions 4 & 5 and the above definition, we deduce that a root product code is full diversity under iterative decoding on both block fading and both erasure channels. The maximal diversity order is reached after one decoding iteration when parallel scheduling is applied, or after two decoding iterations if serial row-column scheduling is performed.

The information rate should not be sacrificed on behalf of diversity. Thus, for $n_c = 2$, we should avoid rate 1/3 codes which are capable of attaining a 3rd order diversity. The product code will be devised according to specific properties.

Design Property 7 *The design coding rate satisfies the following inequalities:*

$$\frac{1}{3} < R \leq \frac{1}{2},$$

or equivalently

$$\frac{\sqrt{3}}{3} < r \leq \frac{\sqrt{2}}{2}.$$

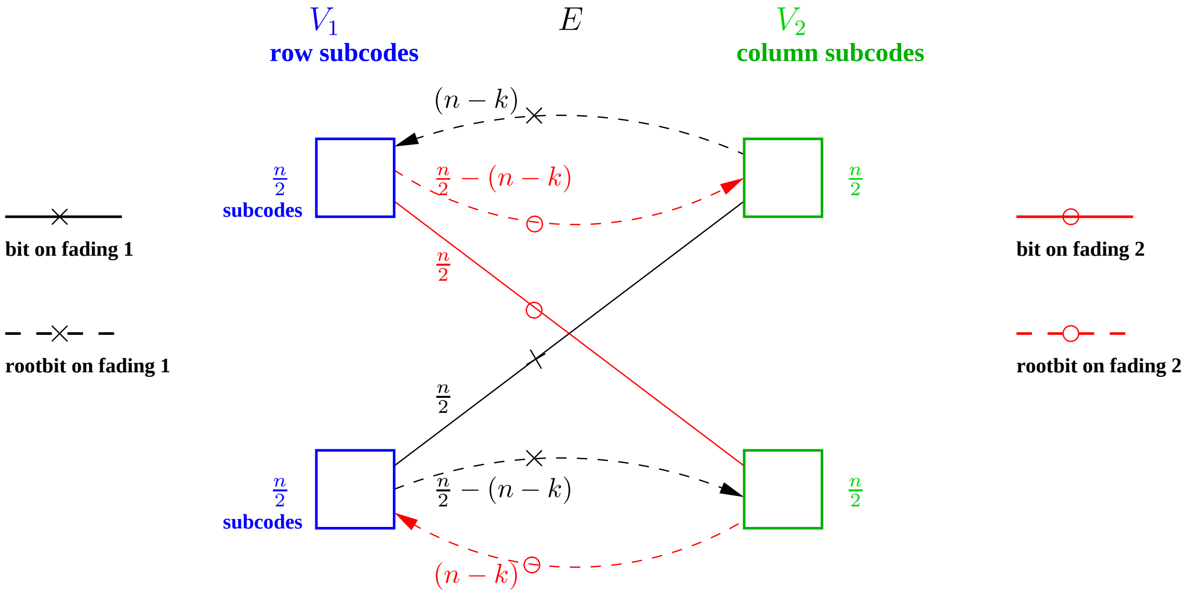

Figure 5: Compact graph of a full-diversity product code with 2 supernodes on each side. Arrows are pointing to rootchecks of the corresponding bit. Graph notations defined in this figure will be used in the sequel. This representation is modified later in order to yield a graph-encodable product code as shown for example in Figures 6 and 8 for $r = \frac{2}{3}$.

A product code $C[N, K] = C_0[n, k]^{\otimes 2}$ is graphically represented by a complete bipartite graph (V_1, V_2, E), where V_1 is the set of row subcodes, V_2 is the set of column subcodes, and E is the set of all bits. The vertices of V_1 and V_2 are completely connected. We also have $|V_1| = |V_2| = n$ vertices and $|E| = n^2$ edges. By convention, row subcodes are drawn on the left and column subcodes on the right. Since we are considering non-ergodic channels with 2 states, V_1 and V_2 are split into two supernodes each containing $\frac{n}{2}$ subcodes. This compact graph representation is depicted in Figure 5. Bits are associated to edges linking row subcodes to column subcodes. Arrows in Figure 5 are pointing to rootchecks of the corresponding bits.

Let us first focus on the upper supernode of V_1. All rows have been assumed to be rootchecks, each of them containing $n - k$ roots as indicated on the edges undergoing fading α_1. Because the graph is complete, the number of edges linking a row supernode and a column supernode must be equal to $\frac{n}{2}$. Hence, there are $\frac{n}{2} - (n-k)$ bits with state α_2 linking the two upper supernodes and $\frac{n}{2}$ bits linking diagonally opposed supernodes from V_1 to V_2, as clearly indicated in Figure 5. The graph structure in the lower supernodes is symmetric to that of upper ones. The graph construction is still valid for odd n and for asymmetric product codes but it should be slightly modified accordingly.
Notice that $\frac{n}{2} - (n-k)$ is a non-negative integer since $\frac{1}{2} < \frac{\sqrt{3}}{3} < r$. Finally, the number of

rootbits K_r is

$$K_r = 2 \times \frac{n}{2} \times \Big((n-k) + \frac{n}{2} - (n-k) \Big) \quad \geq \quad k^2 = K,$$

because $R \leq \frac{1}{2}$. We conclude that all information bits can be covered by rootchecks, i.e., the graph representation given in Figure 5 with 2 supernodes on each side represents a full-diversity product code. We may also be tempted to announce now that a simple solution has been found for the construction of full-diversity root product codes at any rate satisfying the design property 7. Unfortunately, the diagonal links have $\frac{n}{2}$ digits which are parity bits of C. The constituent code C_0 cannot successfully carry out the computation of those parity bits based on the knowledge of information bits because $\frac{n}{2} > (n-k)$ as mentioned above. Therefore, iterative encoding of the product code $C = C_0 \otimes C_0$ based on its row and column subcodes C_0 is impossible when V_1 and V_2 are split into 2 supernodes. In an equivalent terminology, we say that such a structure is not graph encodable.

Design procedure. The compact graph of a graph-encodable root product code is built as follows:

- The number of edges linking two supernodes should be equal to the number of supernodes since the graph is complete.
- To render a graph-encodable code, the number of supernodes cannot be equal to 2 as in the previous graph representation. The number of non-root bits must be less than or equal to $n-k$. In order to let the rootchecks cover all constituent information bits, the number of supernodes in V_1 and V_2 should be taken equal to

  $$\lceil \frac{n}{n-k} \rceil,$$

  i.e., this is the number of supernodes on each side of the compact graph representation, where a supernode does not contain more than $n-k$ subcodes.

- Colors should be selected in order to maximize the number of rootbits. After color selection, information bits are placed on root edges.

Before showing 3 examples of full-diversity product codes obtained via the design procedure described above, we would like to understand better the influence of the compact graph structure on the coding rate. This section is restricted to graphs with strong symmetries where rootbits are decoded in one shot as a consequence of Definition 3. This type of first order rootbits is to be opposed to high order rootbits as defined in the next section, where the compact graph has less symmetries. Unless otherwise stated, the order of a root code is 1 by default. Now, we impose the following design property which is inherited from the initial graph with 2 supernodes on each side.

Design Property 8 *A supernode of V_1 or V_2 has a maximum of 2 super-edges associated to rootbits, one super-edge with incoming rootbits and the other with outgoing rootbits.*

A super-edge is an edge linking two supernodes in the compact graph representation. Our design procedure for first-order root product codes limits the number of binary elements within a superego to a maximum of $n - k$ bits.

Let us assume that all rootchecks are row subcodes, then the code guarantees full diversity for all its information bits if $n(n-k) \geq k^2$, i.e. $r^2 + r - 1 \leq 0$.

Proposition 9 *A full-diversity product code $C_0[n,k]^{\otimes 2}$ with all its rootchecks being row subcodes satisfies $r = \frac{k}{n} \leq \frac{\sqrt{5}-1}{2}$.*

Such an unbalanced code should be avoided in practice. The constituent coding rate should be lower bounded by $(\sqrt{5}-1)/2$ instead of $\sqrt{3}/3$ as stated in the design property 7. Now, let us examine encoding on the subcode level for a non-ergodic channel with 2 states. A root product code is graph-encodable if the subcode dimension does not exceed the number of incoming and outgoing roots, i.e., if $k \leq 2(n-k)$.

Proposition 10 *A graph-encodable full-diversity root product code exists if $r = \frac{k}{n} \leq \frac{2}{3}$.*

From the above propositions, we conclude that a valid range for the rate of the product code constituent is

$$\frac{\sqrt{5}-1}{2} \leq r \leq \frac{2}{3}.$$

Corollary 11 *A graph-encodable full-diversity root product code $[16,11,4]^{\otimes 2}$ does not exist.*

Of course, this corollary is meant for first order root codes. A third order full-diversity $[16,11,4]^{\otimes 2}$ code is built in the next section. Finally, we end this section by illustrating our design procedure in Figures 6-11 with square product codes based on linear binary constituents $[6,4]$, $[12,8]$, and $[5,3]$.

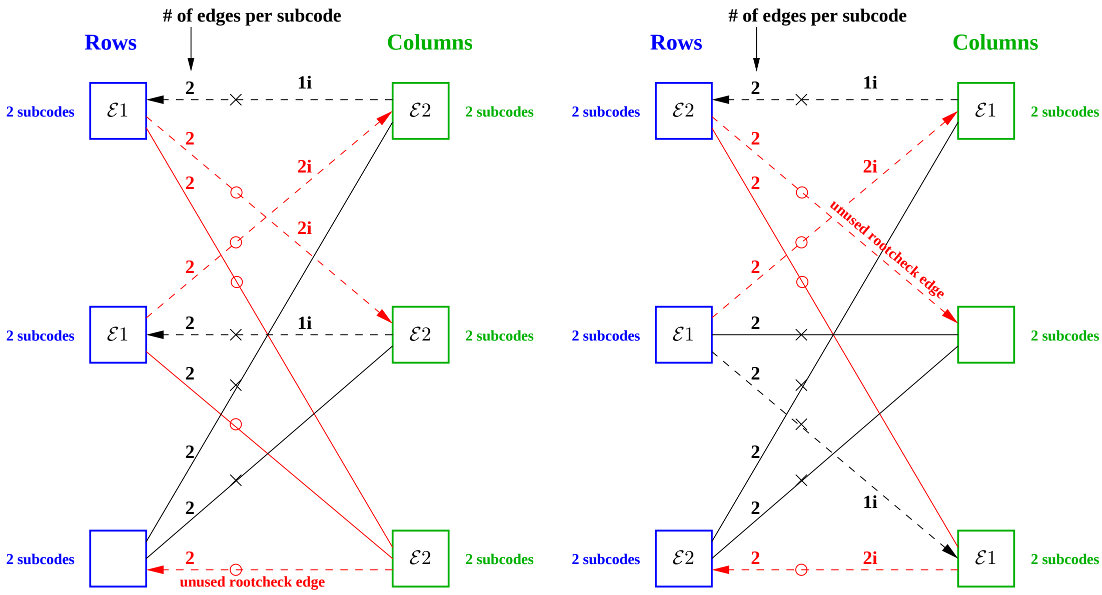

Figure 6: Two compact graph representations of a full-diversity product code $[6,4,2]^{\otimes 2}$, $r = \frac{2}{3}$ and $R = \frac{4}{9} = 0.4444$. Information bits transmitted on fading 1 (resp. fading 2) are indicated on the graph edges by $1i$ (resp. $2i$). The encoding can be made via the graph using the activation schedule $\mathcal{E}1$ followed by $\mathcal{E}2$. Any channel state (α_1 or α_2) can be assigned to unused rootcheck edges.

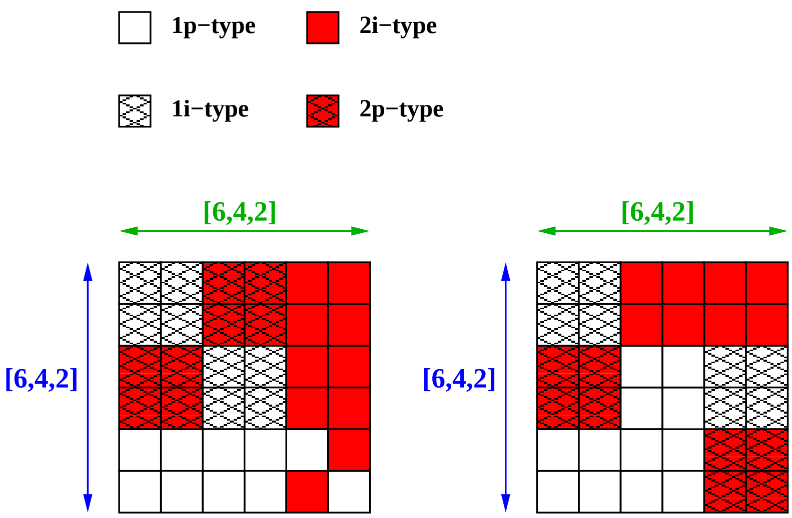

Figure 7: Matrix representations of the full-diversity product codes $[6,4]^{\otimes 2}$ defined by the compact graphs in Figure 6.

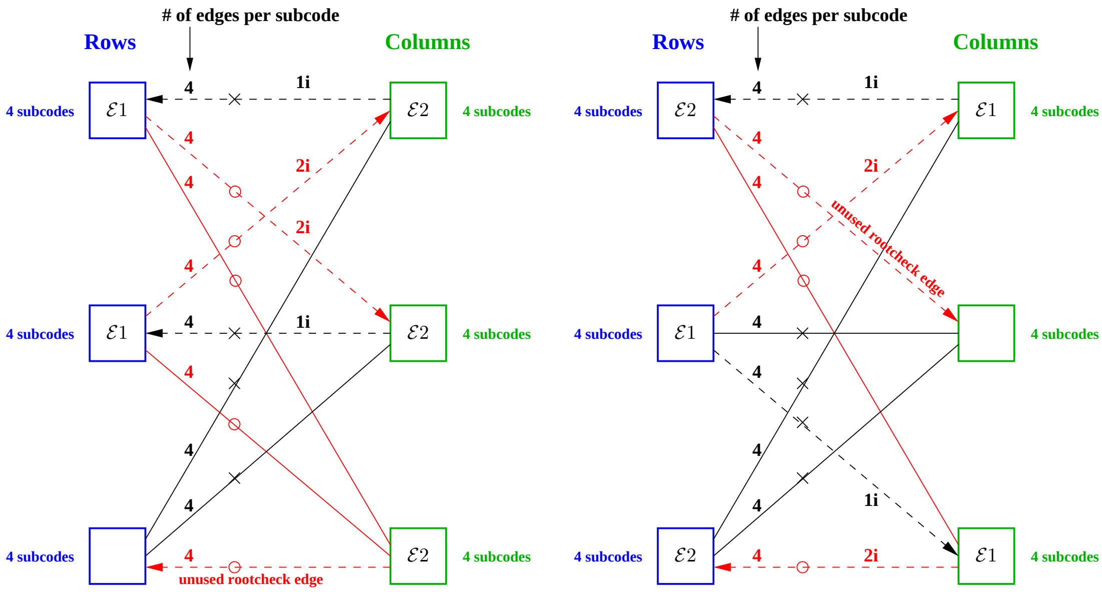

Figure 8: Two compact graph representations of a full-diversity product code $[12, 8, 3]^{\otimes 2}$, $r = \frac{2}{3}$ and $R = \frac{4}{9} = 0.4444$. Information bits transmitted on fading 1 (resp. fading 2) are indicated on the graph edges by $1i$ (resp. $2i$). The encoding can be made via the graph using the activation schedule $\mathcal{E}1$ followed by $\mathcal{E}2$. Any channel state (α_1 or α_2) can be assigned to unused rootcheck edges.

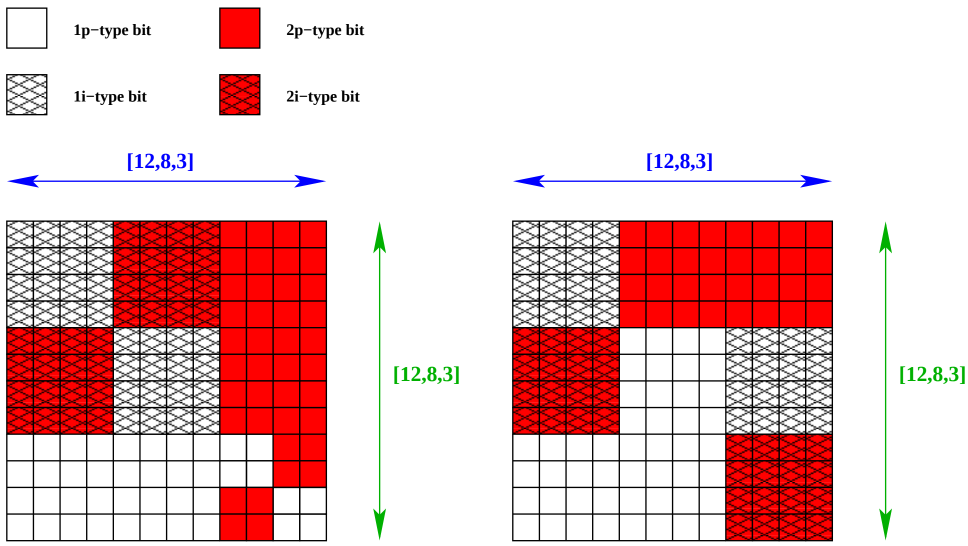

Figure 9: Matrix representations of the full-diversity product codes $[12, 8]^{\otimes 2}$ defined by the compact graphs in Figure 8. All matrix patterns with $r = \frac{2}{3}$ are identical, as in Figure 7.

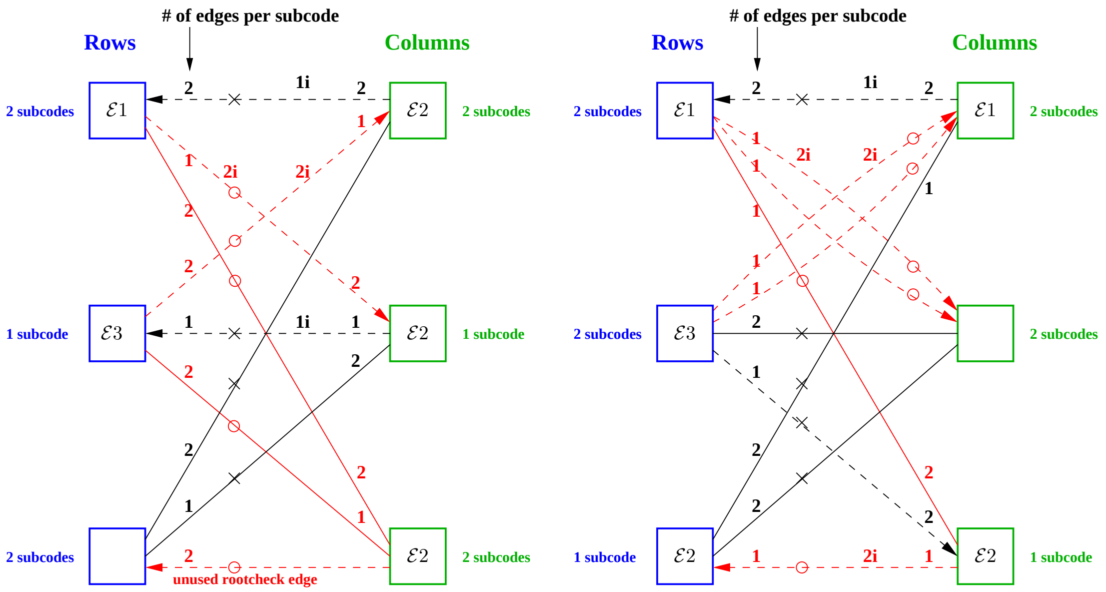

Figure 10: Two compact graph representations of a full-diversity product code $[5, 3, 2]^{\otimes 2}$, $r = \frac{3}{5}$ and $R = \frac{9}{25} = 0.36$. Since the subcode $[5, 3]$ is a shortened version of $[6, 4]$, this product graph is obtained by shortening the graph in Figure 6. The compact representation may have its limitation: the encoding schedule is different on the right. Also, not all parallel edges connected to a rootcheck are considered as information bits since $k = 3$.

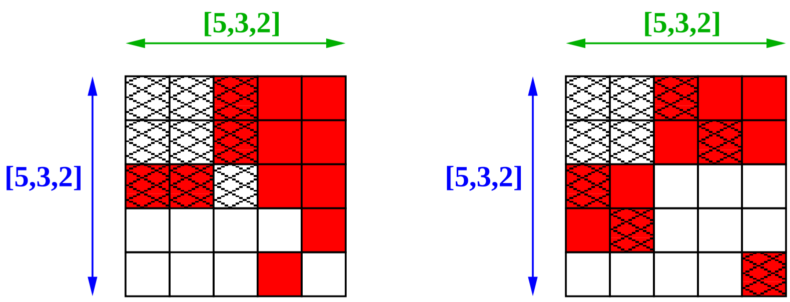

Figure 11: Matrix representations of the full-diversity product code $[5, 3]^{\otimes 2}$ defined by the compact graphs in Figure 10.

6 Product Codes with High-Order Rootchecks

Is it still possible to find a full-diversity product code under iterative decoding when the number K_r of rootbits is less than k^2? From the study made in the previous section, the answer is no due to the strong constraint stating that all information bits are decoded at full-diversity after one decoding iteration only. Assume that $K_r < k^2$. Can full diversity be achieved beyond the first iteration and how many decoding iterations should be performed? The answer to those supposedly difficult questions can be elucidated via high order rootbits. This situation is well clarified by the tree structure in Figure 12. Since the girth in the product code graph is equal to 4, order-4 rootbits do not exist. Assume that the root is colored in white. For $\rho \leq 2$, this root has order $\rho + 1$ if the leaves are red of any order or white with order ρ. The red leaves can be rootbits or non-rootbits, their type does not modify the rootbit order.

Proposition 12 *A root product code attains full diversity order (or equivalently recovers all information bits) after at most 3 decoding iterations.*

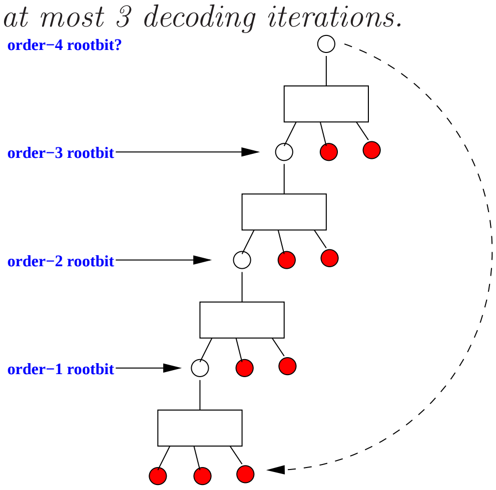

Figure 12: Tree structure for defining rootbits of order $\rho = 1, 2, 3$. By convention, a bit that never achieves full diversity under iterative decoding will be assigned $\rho = +\infty$.

Now, let us describe the construction of the full-diversity product code $[16, 11]^{\otimes 2}$. Firstly, we build a bipartite compact graph with $\lceil \frac{n}{n-k} \rceil = \lceil 16/5 \rceil = 4$ supernodes V_{1i} on the left and 4 supernodes V_{2i} on the right, $i = 1 \ldots 4$, where $V_j = \bigcup_i V_{ji}$, $|V_{j1}| = |V_{j2}| = |V_{j4}| = n - k = 5$ and $|V_{j3}| = n \bmod n - k = 1$, $j = 1, 2$. This arrangement of supernodes aims at positioning the $k \times k$ information bits in the upper left corner in the bi-dimensional matrix representation. Figure 13(a) depicts the matrix form of the product code at step 1. The location of rootbits can be easily derived, e.g., edges $V_{11} \leftrightarrow V_{21}$ are rootbits associated to the 5 rootchecks defined by the subcodes in V_{11}. Similarly, the 5 edges $V_{12} \leftrightarrow V_{21}$ are rootbits associated to V_{21}. Figure 13(a) also shows that V_{23} is the rootcheck of 3 bits, one connected to $V_{1,3}$ and 2 connected to V_{14}. The total number of rootbits in this initial configuration is maximized and is equal to 136 bits (greater than 11^2). Unfortunately, if you cover the product code information bits by rootbits, it can be easily verified that iterative

row-column encoding is impossible. Hence, Figure 13(a) presents a full-diversity product code which is not graph-encodable. The graph-encodable root product code is obtained after two more steps: **Step 2:** Color in red the 3 white bits in V_{23} and color in white the 5 bits $V_{23} \leftrightarrow V_{12}$. The result of this operation is that the first 11 bits of V_{23} are all full diversity. Their order ρ can be observed in Figure 15. The 25 bits $V_{12} \leftrightarrow V_{22}$ are still full diversity but they degraded from $\rho = 1$ to $\rho = 2$. **Step 3:** Color in red 5 bits located on a diagonal on edges $V_{14} \leftrightarrow V_{21}$. The result of this operation is that the first 11 bits of V_{13} are all full diversity. Step 4 is meaningless, its role is to delete some red bits (color them in white) in order to balance the number of bits transmitted on the 2 channel states. The final structure is given in Figure 13(d).
We end the paper by proposing a $[20, 15, 3] \otimes [15, 10, 4]$ product code which is graph-encodable, full-diversity, and achieving the highest possible rate $R = \frac{1}{2}$, suitable for both block erasure and block fading channels. The design follows the procedure given in the previous section but the graph is asymmetric since the product code is rectangular. Figures 16, 17, and 18 present the main properties of this full-diversity product code.

7 Conclusions

A finite-length design of bi-dimensional binary product codes suitable for block fading channels has been proposed. The study is based on graphical tools and some simple algebraic properties of product codes. Codes at several coding rates capable of achieving the highest diversity order have been found. This work should be enhanced via the analysis of the coding gain and the general asymptotic performance behavior of product codes on non-ergodic channels.

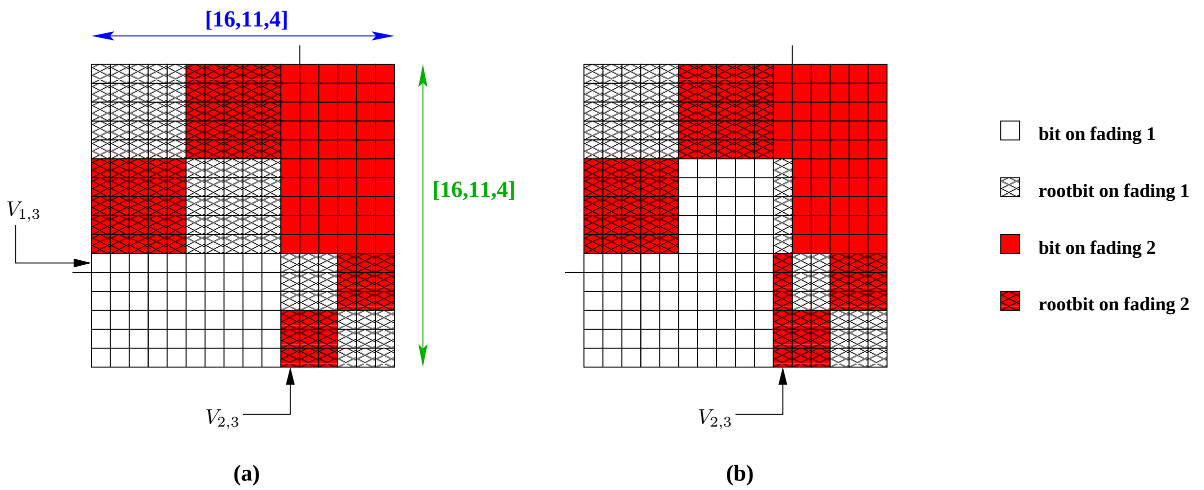

Figure 13: Steps 1 and 2 in the construction procedure of $[16, 11]^{\otimes 2}$. Dashed white and red rootbits have order $\rho = 1$.

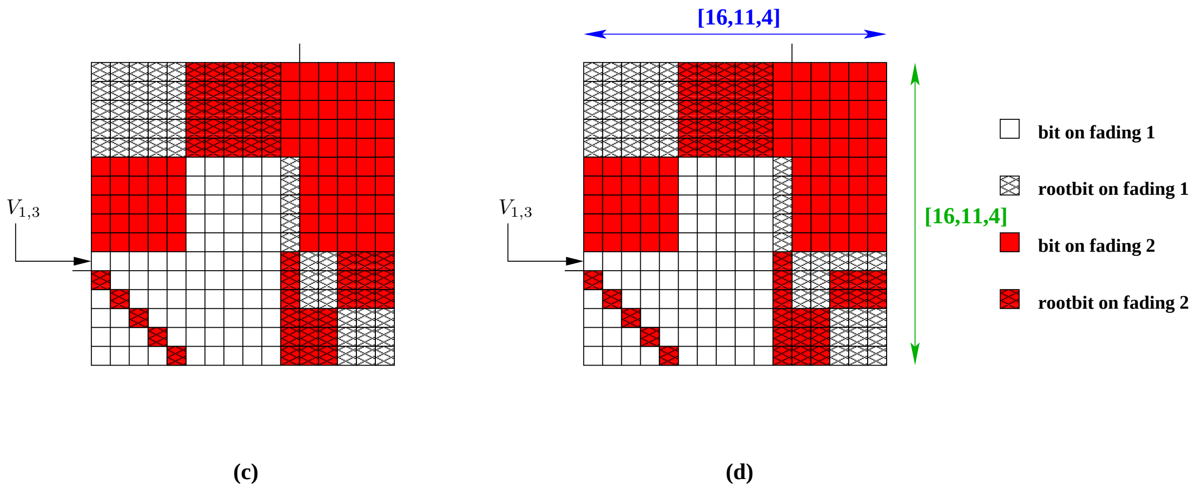

Figure 14: Steps 3 and 4 in the construction procedure of $[16,11]^{\otimes 2}$. Dashed white and red rootbits have order $\rho = 1$.

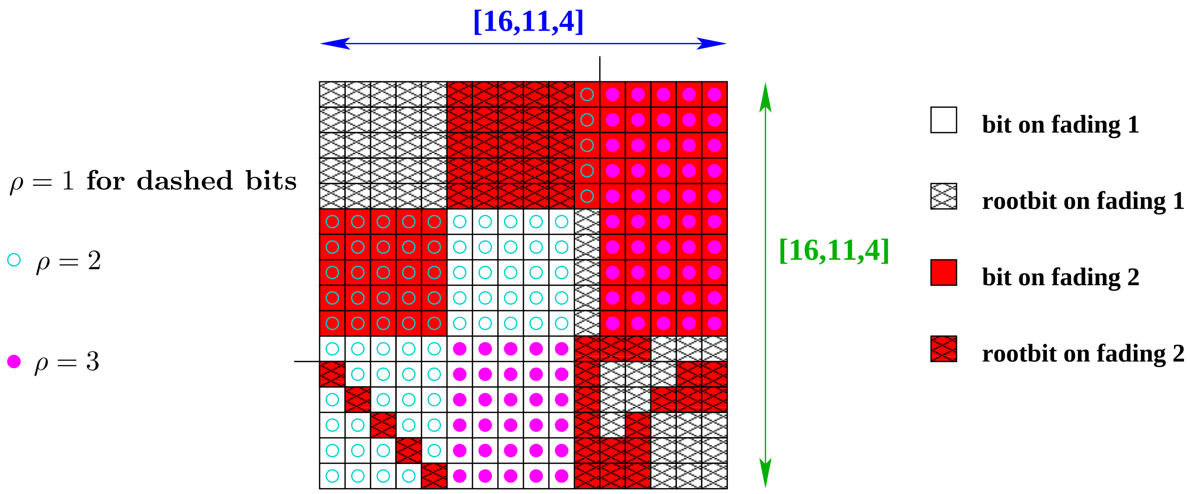

Figure 15: Full-diversity root product code $[16,11,4]^{\otimes 2}$ of order 3. Overall coding rate is $R = 0.4726$, 3 iterations are needed to attain full diversity. For a given bitnode, ρ iterations are needed to reach full diversity under parallel scheduling. Two red bits have moved to $V_{1,3}$ in order to match the code in Figure 14(d) to the specific $[16,11,4]$ version we used in our experimental results.

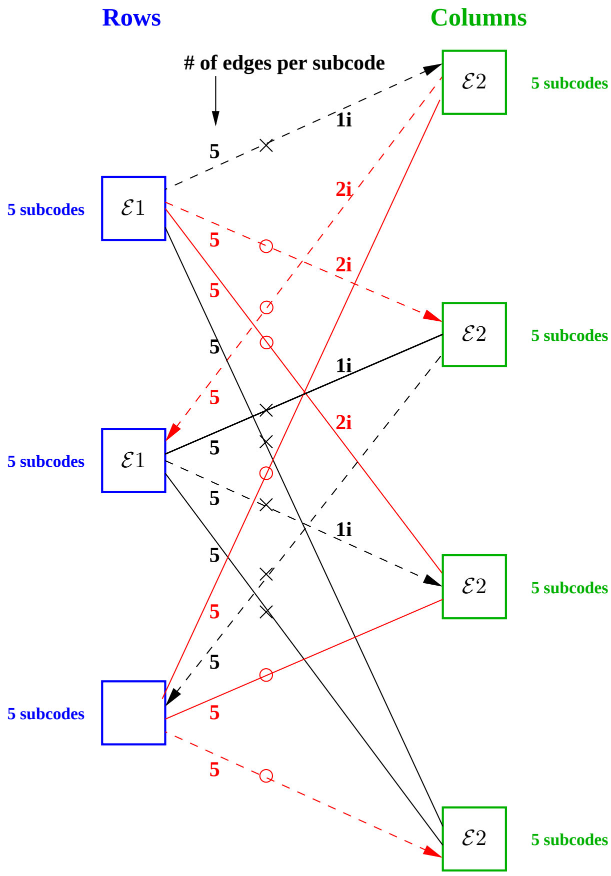

Figure 16: A compact graph representation of a full-diversity product code $[20, 15, 3] \otimes [15, 10, 4]$, coding rate is $R = \frac{1}{2}$. Information bits transmitted on fading 1 (resp. fading 2) are indicated on the graph edges by $1i$ (resp. $2i$). The encoding can be made via the graph using the activation schedule $\mathcal{E}1$ followed by $\mathcal{E}2$.

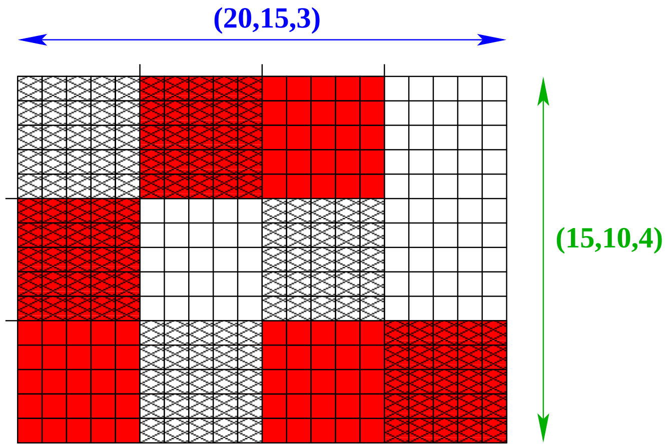

Figure 17: Matrix representation of a full-diversity product code $[20, 15, 3] \otimes [15, 10, 4]$ with high order rootchecks, overall coding rate is $R = \frac{1}{2}$. Information bits are located in the standard 10×15 upper left corner.

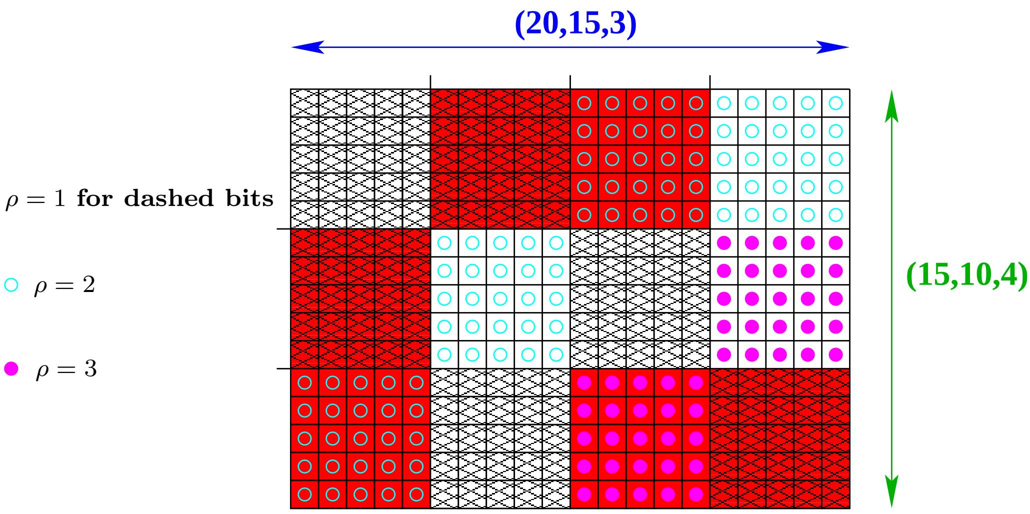

Figure 18: Matrix representation of a full-diversity product code $[20, 15, 3] \otimes [15, 10, 4]$ with high order rootchecks, where the order ρ of each bit is indicated.

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