Iterative Processing of Information

Joseph J. Boutros

Ecole Nationale Supérieure des Télécommunications, Paris

AUST, Lebanon

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Presentation outline

1. Historical notes and motivation
2. Codes on graphs for the erasure channel
3. Asymptotic analysis of iterative decoding
4. Probabilistic decoding on soft channels (e.g. the gaussian channel)
5. Iterative receivers for multiple access (CDMA) and multiple antennas (MIMO)

Introduction

Motivation:

1. Split a complex problem into two or more simpler sub-problems.
2. Under a given optimality criterion, aim at finding the optimal solution with a reasonable complexity.
3. Iterative methods are useful for both design (e.g. compound codes) and optimization (e.g. iterative decoding).

Where?

1. Iterative methods exist in numerical analysis and other fields in mathematics.
2. They appeared in the field of digital transmission at least since the mid 50's.
   - In coding and information theory
   - In communications theory
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Key Dates & Papers (1/2)

All ingredients are here for the birth of modern coding/communication theory.

- P. Elias, Product codes and iterative decoding, 1954.
- Lloyd and MacQueen, The k-means algorithm, 1957-1967.
- R. Gallager, Low-Density Parity-Check (LDPC) codes, 1962.
- Baum, Petrie, Soules, and Weiss, The Baum-Welch algorithm, 1970.
- Blahut & Arimoto, Computation of channel capacity, 1972.
- Bahl, Jelinek, Coke, and Raviv, The forward-backward algorithm, 1974.
- Hartmann & Rudolph, APP decoding based on the dual code, 1976.
- Dempster, Laird, and Rubin, The Expectation-Maximization algorithm, 1977.
- G. Battail, Replication decoding & Soft output Viterbi, 1979-1987.
- M. Tanner, Graph codes and iterative algorithms, 1981.
- J. Pearl, Probabilistic reasoning in intelligent systems, 1988.

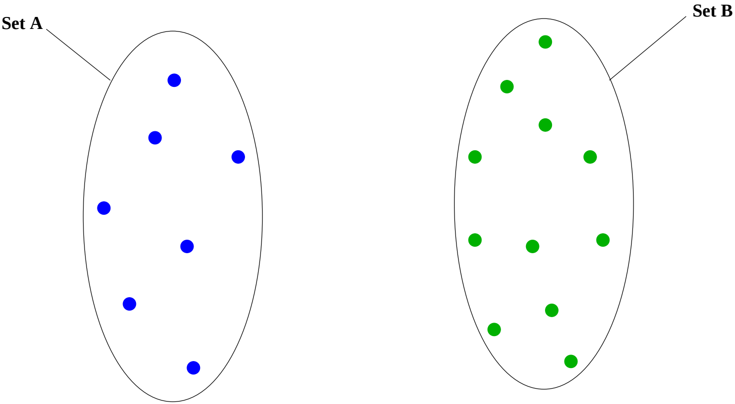
Key Dates & Papers (2/2)

The era of capacity achieving codes.

- Berrou & Glavieux, Parallel Turbo Codes, 1993.
- D. Mackay, Rediscovery of LDPC codes and new improvements, 1995.
- Luby, Mitzenmacher, Shokrollahi, Spielman, and Stemann, Irregular LDPC codes, 1997.
- More recent papers (1998-2005) can be found in IEEE journals and other scientific publications.

Toy Example

- Alternate optimization by Csiszár & Tusnady (1984).
  Toy example: Find the minimum distance between two sets A and B.



- Exhaustive search: $|A| \times |B| = O(n^2)$ metric computations. It is possible to find the minimum distance after $O(n)$ metric computations.

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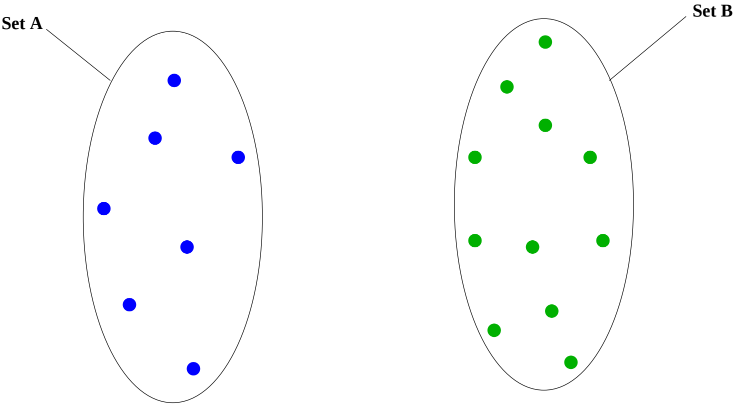

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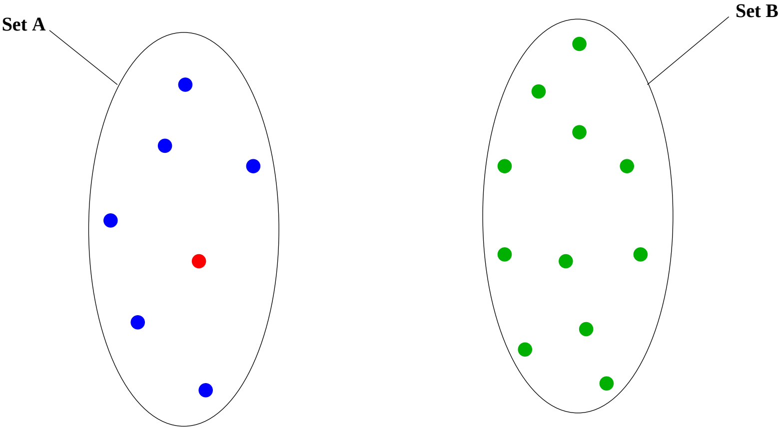
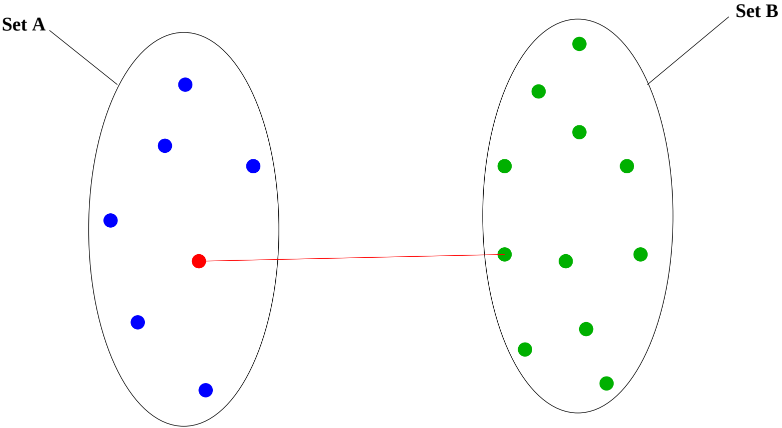
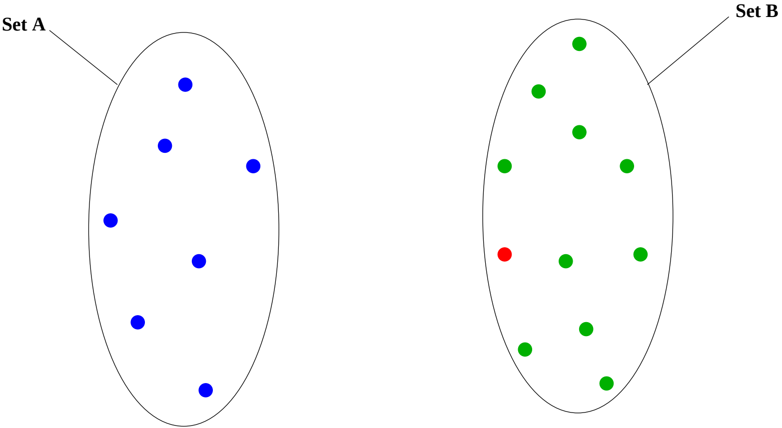


Iteration 1 - Minimize distance



Iteration 2 - Initialize



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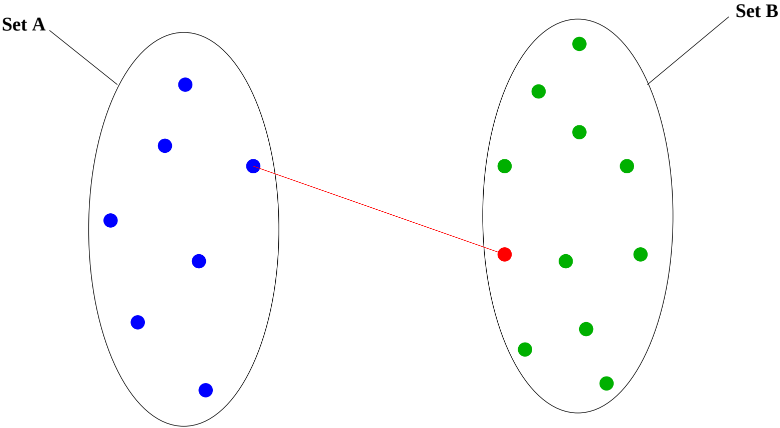

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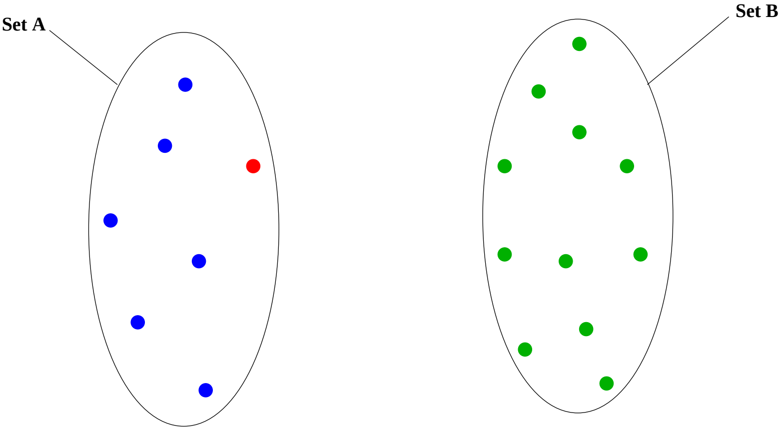

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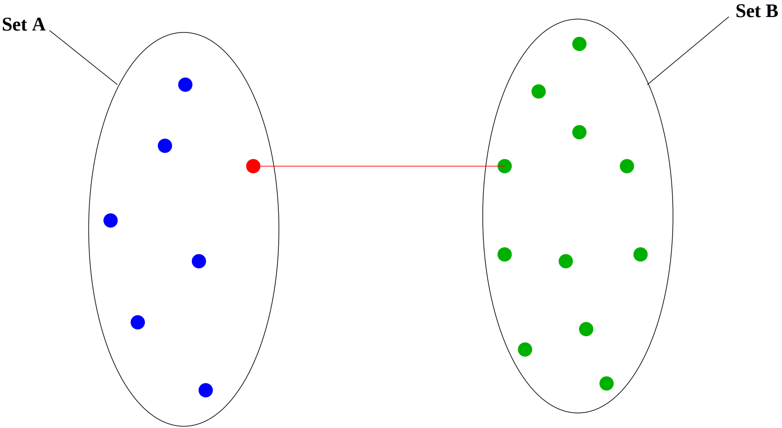
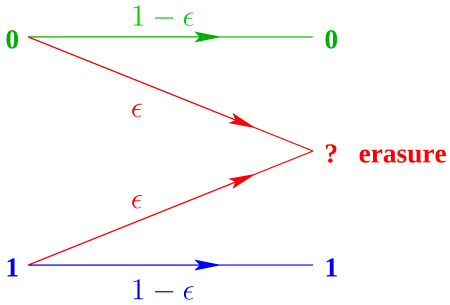


Codes on graphs for the erasure channel

We consider the ergodic binary erasure channel (BEC). The channel input is binary and its output is ternary. Only erasures occur on this channel, no erros.



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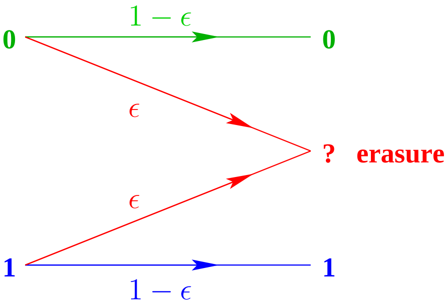

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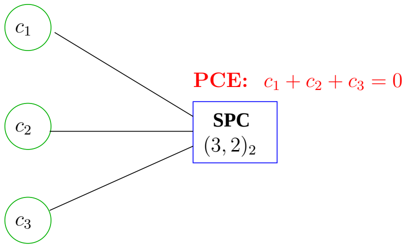
Bipartite graph representation, the Tanner Graph (1/2)

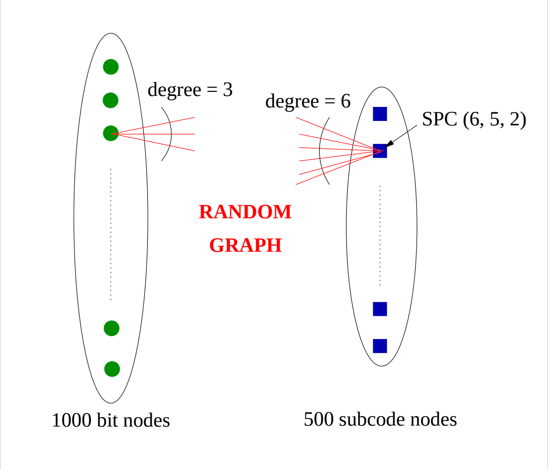
- We build a code defined by a graph in order to fill erasures on the BEC.
- A low-density parity-check (LDPC) code of length N bits and dimension K is defined by a bipartite graph.
- Bitnodes are drawn as circles, checknodes are drawn as squares.



- For a regular (d_b, d_c) LDPC, bitnodes have degree d_b and checknodes have degree d_c. The coding rate is

$$R_c = \frac{K}{N} \geq 1 - \frac{d_b}{d_c}$$

Bipartite graph representation, the Tanner Graph (2/2)



Tanner graph of an LDPC code, length $N = 1000$, dimension $K = 500$, coding rate $R_c = 1/2$. The 3000 edges are chosen at random.

Decoding algorithm

The number of checknodes is denoted by $L = N - K$.
In the presence of a uniform source encoded by an LDPC(N,K) code whose codewords are transmitted on an iid BEC channel, the iterative non probabilistic decoding algorithm is given by the following steps:

- Step 0: Initialize $Iter = 0$ and $j = 1$.
- Step 1: Count the number μ of erased bits connected to checknode j.
- Step 2: If $\mu = 1$ then fill the erased bit by summing other bits modulo 2.
- Step 3: Increment j. If $j > L$ then increment $Iter$ and set $j = 1$.
- Step 4: If $Iter > MaxIter$ then Stop else Goto Step 1.

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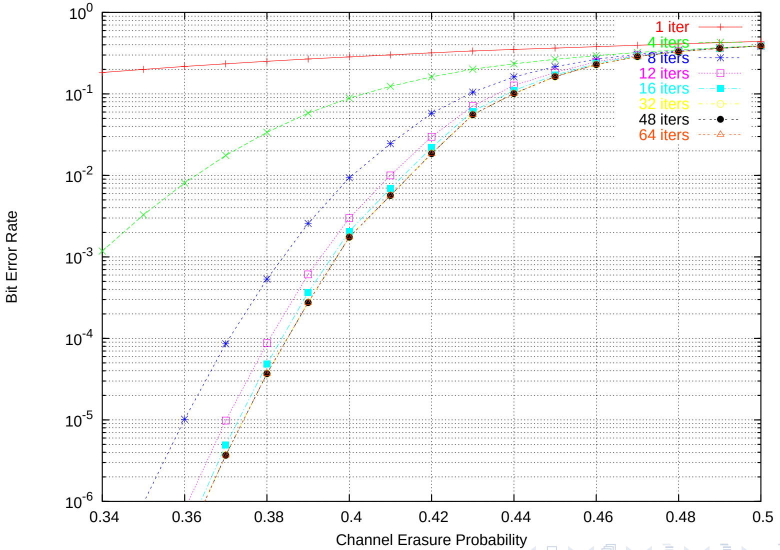
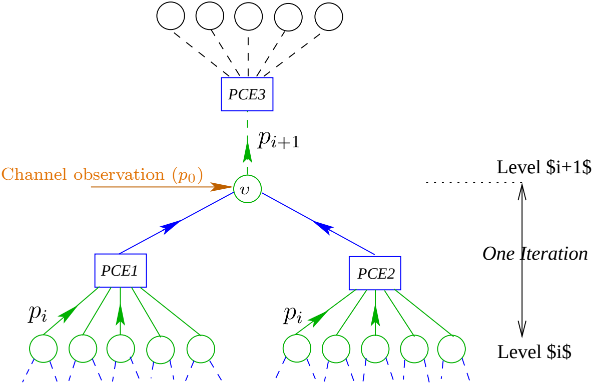
Regular (3,6) binary LDPC, $N = 1000$, $K = 500$, iid BEC



Analysis of iterative decoding (1/3)

Assume an infinite length code with a graph representation without cycles (tree representation). The local neighborhood of a bitnode v in a regular (3,6) LDPC code is shown below. Let p_i denote the erasure probability at iteration i.



It is trivial to prove that

$$p_{i+1} = p_0 \times \left(1 - (1 - p_i)^5\right)^2$$

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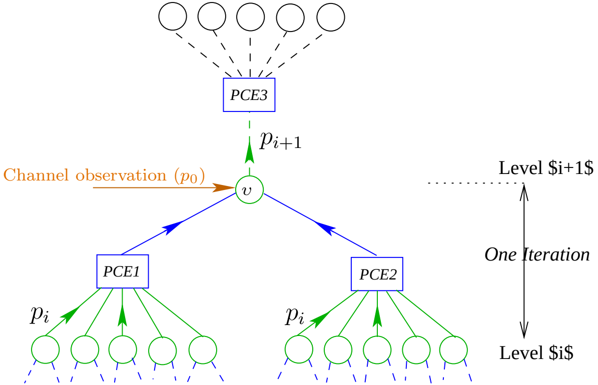

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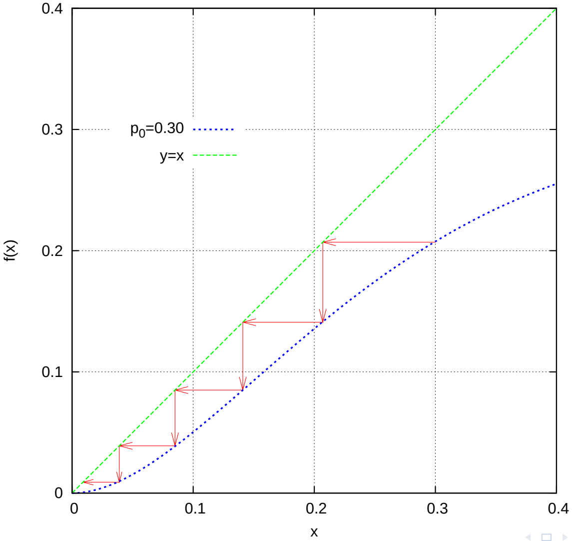
Analysis of iterative decoding (2/3)

Plot of the transfer function $f(x) = p_0(1-(1-x)^5)^2$ (in blue) versus $y = x$ (in green).



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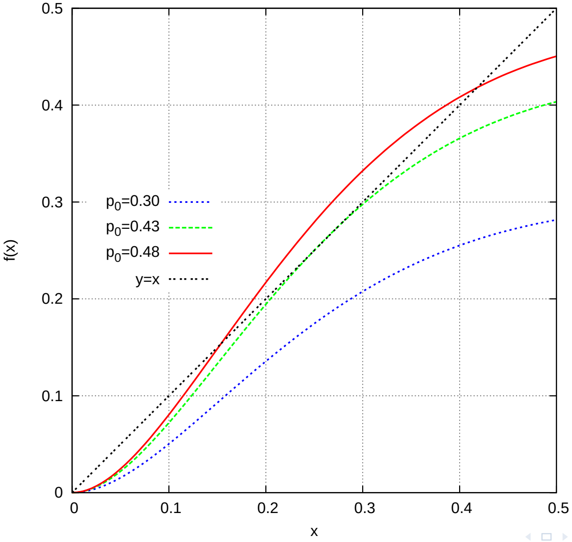

Evolution for irregular codes (1/2)

- An LDPC code is said to be 'irregular' if graph nodes of same kind do not have equal degree.
- Left irregularity: Bitnodes have different degrees. The fraction of edges connected to bitnodes of degree i is λ_i. The degree distribution is defined by

$$\lambda(x) = \sum_{i=1}^{d_b} \lambda_i x^{i-1}$$

where $\lambda(1) = 1$ and $0 \leq \lambda_i \leq 1 \ \forall i$.
- Right irregularity: Checknodes have different degrees. The fraction of edges connected to checknodes of degree j is ρ_j. The degree distribution is defined by

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- By looking to the tree neighborhood of a graph node, it is easy to show that erasure probability p_i after LDPC decoding at iteration i on the binary erasure channel satisfies
$$p_{i+1} = p_0 \times \lambda(1 - \rho(1 - p_i))$$
where $p_0 = \epsilon$ is the channel erasure probability.
- The stability condition of the fixed point at the origin is obtained by writing that $f'(0) < 1$. Thus, the necessary and sufficient condition for stability of 0 is
$$\lambda'(0) \times \rho'(1) \times p_0 < 1$$
- For a general type of channels, assume that all-zero codeword is transmitted (geometrical uniformity is satisfied). If log-ratios are used, $LR = \log\left(\frac{P(0)}{P(1)}\right)$, then the fixed point is at $+\infty$. The stability condition becomes
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APP Decoding of Binary SPC Codes (1/2)

Let us describe the soft-input soft-output (SISO) decoder which is capable of determining the a posteriori probability $APP(c_i) = P(c_i|r, PCE)$ for each bit c_i.

- The observation of c_i is a probability proportional to the channel likelihood

$$obs(c_i) \propto p(r_i|c_i) \propto exp(-\frac{(r_i - I(c_i))^2}{2\sigma_n^2})$$

- The a priori information $\pi(c_i)$ is produced by a genie. Any other code connected to our code is a potential genie. Write $\pi(c_i) = 1/2$ when no a priori information is available.
- The extrinsic information is generated by the SISO decoder based on the PCE constraint. It can be considered as a new a priori information produced by our decoder. For example, since $c_1 = c_2 + c_3 + \ldots + c_n$, the SISO-APP decoder computes $Extr(c_1)$ as follows:

$$Extr(c_1 = 1) = \frac{1}{2} \times \left(1 - \prod_{i=2}^{n}(1 - 2p_i)\right)$$

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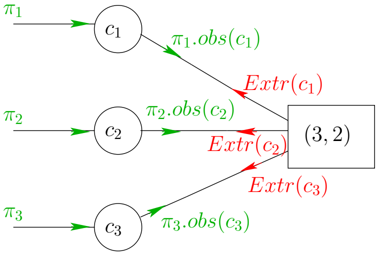
APP Decoding of Binary SPC Codes (2/2)

- The a posteriori probability, given the total observation r, some apriori information π_i and the SPC code constraint, is proportional to the product of the two opposite streams on a graph edge,

$$APP(c_i) \propto \pi_i \times obs(c_i) \times Extr(c_i)$$

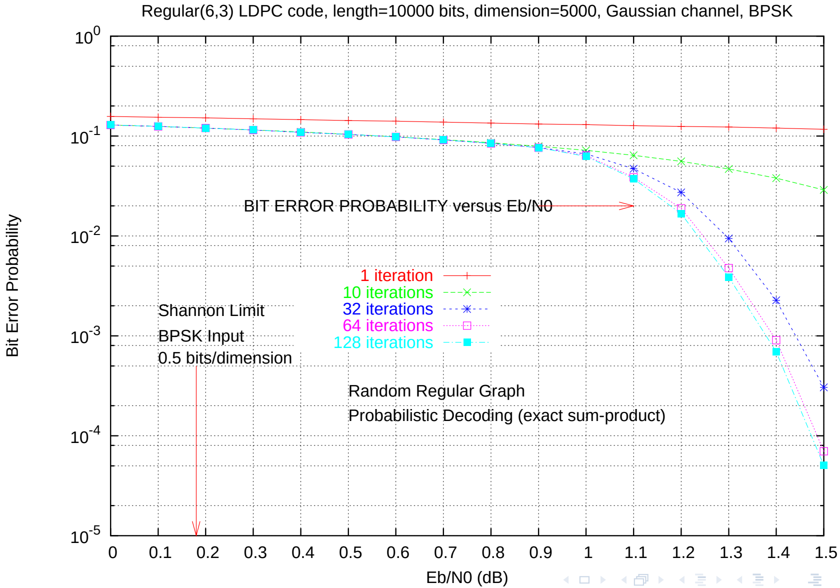
Information Propagation from left to right



Information Propagation from right to left

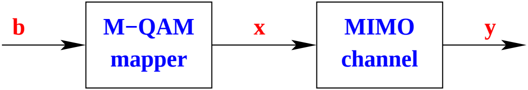
- The proportionality factors are determined by forcing the following sum: $APP(c_i = 0) + APP(c_i = 1) = 1$.

(3,6) LDPC, $N = 10000$, $K = 5000$, BPSK + AWGN



Iterative APP detection for MIMO channels (1/2)

Consider a frequency non-selective multiple-input multiple-output (MIMO) channel with n_t transmit antennas and n_r receive antennas.



- Channel coefficients are given by the entries of a $n_t \times n_r$ matrix $H = [h_{ij}]$, where h_{ij} is the complex fading of the channel path linking transmit antenna i to receive antenna j.

- The channel output is

$$\mathbf{y} = \mathbf{x}H + \eta$$

  where $\mathbf{x} \in (M - QAM)^{n_t} \subset \mathbb{C}^{n_t}$, $\mathbf{y} \in \mathbb{C}^{n_r}$, and η is an additive white complex gaussian noise vector.

- A multidimensional alphabet $\Omega = (M - QAM)^{n_t}$ of size $M^{n_t} = 2^{mn_t}$.

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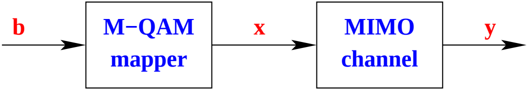

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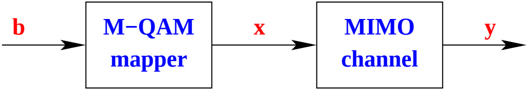

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Iterative APP detection for MIMO channels (2/2)

- The observation of a multidimensional symbol is

$$obs(\mathbf{x}) \propto p(\mathbf{y}|\mathbf{x}) \propto exp\left(-\frac{\|\mathbf{y}-\mathbf{x}H\|^2}{2\sigma^2}\right)$$

- Independent a priori information $\pi(\mathbf{x}) = \prod_{j=1}^{mn_t} \pi(b_j)$.
- The a posteriori probability for a complex QAM symbol is

$$APP(x_i) = \sum_{\mathbf{x}\in\Omega|x_i} APP(\mathbf{x}) \propto \sum_{\mathbf{x}\in\Omega|x_i} obs(\mathbf{x}) \prod_{\ell=1}^{n_t} \pi(x_\ell) \propto \pi(x_i)\ Extr(x_i)$$

- The a posteriori information for binary elements is

$$APP(b_j) \propto \pi(b_j) \underbrace{\sum_{\mathbf{x}\in\Omega|b_j} obs(\mathbf{x}) \prod_{\ell\neq j} \pi(b_\ell)}_{Extr(b_j)}$$

Iterative APP detection for MIMO channels (2/2)

- The observation of a multidimensional symbol is

$$obs(\mathbf{x}) \propto p(\mathbf{y}|\mathbf{x}) \propto exp\left(-\frac{\|\mathbf{y}-\mathbf{x}H\|^2}{2\sigma^2}\right)$$

- Independent a priori information $\pi(\mathbf{x}) = \prod_{j=1}^{mn_t} \pi(b_j)$.
- The a posteriori probability for a complex QAM symbol is

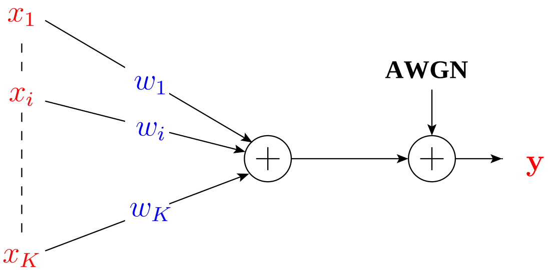
$$APP(x_i) = \sum_{\mathbf{x}\in\Omega|x_i} APP(\mathbf{x}) \propto \sum_{\mathbf{x}\in\Omega|x_i} obs(\mathbf{x}) \prod_{\ell=1}^{n_t} \pi(x_\ell) \propto \pi(x_i)\ Extr(x_i)$$

- The a posteriori information for binary elements is

$$APP(b_j) \propto \pi(b_j) \underbrace{\sum_{\mathbf{x}\in\Omega|b_j} obs(\mathbf{x}) \prod_{\ell\neq j} \pi(b_\ell)}_{Extr(b_j)}$$

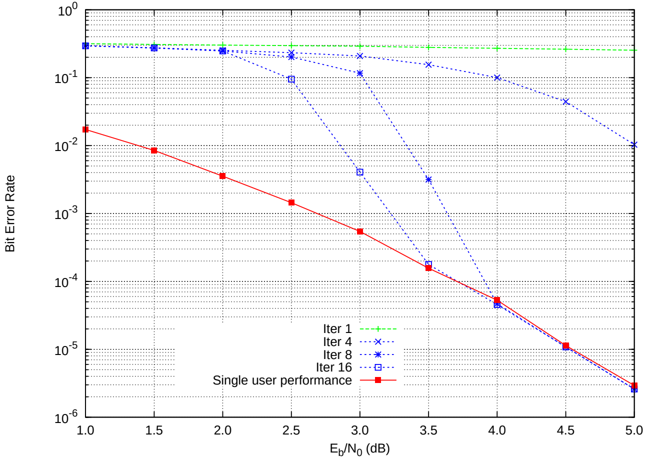
Iterative APP detection for CDMA (1/2)

Consider a chip synchronous code division multiple access channel for K users.



- Symbols x_i belong to a linear modulation, e.g. $x_i = \pm 1$.
- Channel gain coefficient for user i is ω_i.
- Equations for APP multiuser (CDMA) detection are similar to those encountered in MIMO detection.

Iterative APP detection for CDMA (2/2)



Iterative APP joint detection in CDMA with $K = 4$ users on a gaussian channel. The NRNSC code is a rate $1/4$ 16-states (25,27,33,37) for all users. Same SNR per bit for all users. Each user pseudo-randomly interleaves its $N = 8192$ bits before transmitting on the multiple access channel. No PN spreading. System load is 100%.

Conclusions

- Concatenating simple elementary codes leads to powerful compound codes.
- Iterative decoding/detection is an efficient tool in compound coding systems.
- Iterative processing of information opens a new era in coding and communications.
- Future telecommunications products will benefit from iterative probabilistic processing in order to boost performance and to gain in flexibility and compatibility.

Supplementary references have been cited by the speaker during the talk.