Non-Uniform Spatial Coupling

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***Abstract*—A new method for spatial coupling of low-density parity-check ensembles is proposed. The method is inspired from overlapped layered coding. Edges of local ensembles and those defining the spatial coupling are separately built. The new method allows the construction of non-uniform coupling chains with near-Shannon spatially-varying thresholds under iterative decoding. A direct application of non-uniform spatial coupling is unequal error protection of information.**

I. Introduction

Linear block codes and convolutional codes were developed by two separate communities as witnessed in the history of channel coding. Viterbi and Omura reported a higher error exponent for time-varying convolutional codes, see section 5.2 in [1]. Another major success of convolutional codes is found in the construction of parallel turbo codes [2] and its counterpart is found in [3] for low-density parity-check (LDPC) codes. In recent coding theory, both linear block codes and convolutional codes are studied or represented in similar ways leading to new structures such as LDPC convolutional codes [4], Tanner convolutional codes [5], and quasi-cyclic block and convolutional codes [6]. Exceptional thresholds of LDPC convolutional codes [7] [8] led to the discovery of threshold saturation [9] and to the general concept of spatial coupling [10]. Spatial coupling is currently applied in many areas in coding and information theory. The wide application of spatial coupling ranges from coding for the wiretap channel [11] to coding for the multiple access channel [12]. A long list of applications can be found in [10]. Generally coupled structures including multiple chains and loops have been proposed by Truhachev et al. [13] [14]. Their connected spatially-coupled chains aim at improving the iterative decoding threshold and minimizing the decoding complexity.

A spatially-coupled LDPC ensemble is formed by multiple LDPC ensembles sharing common edges. The whole coupled structure resembles a convolutional code with extra check nodes on both extremal sides [9] playing the role of a trellis termination. The parity-check description of a coupled ensemble is similar to LDPC convolutional codes originally proposed by Felström and Zigangirov [4]. Trellis termination for a chain of coupled ensembles has a direct impact on threshold saturation and on the finite-length performance. Chain loops is another way of improving the threshold instead of terminating the chain on both sides [13]. For asymptotic coding schemes, the advantages of spatially-coupled ensembles can be recapitulated in three main points: regular degree distributions substituting for complex irregular distributions, iterative belief propagation (BP) decoding achieving maximum-a-posteriori (MAP) performance, and a universal spatial coupling concept that can be applied in many areas beyond LDPC coding.

In this paper, we propose a new method for spatial coupling. Firstly, our method considers a chain of uncoupled LDPC ensembles, then new edges are added to the chain to couple an ensemble with its neighbor on one side. This method, referred to as forward layered LDPC coupling, is inspired from unequal error protection (UEP) and layered coding. Indeed, our initial objective was to create a limited number of classes for UEP [15] [16] [17], e.g. three classes of digits with decreasing error rate performance to imitate graceful degradation encountered in analog systems. UEP can be carried out via layered coding, a well-known technique for video coding [18] [19] [20]. Layered coding has been recently proposed for the Gaussian interference channel [21]. A large number of levels in layered coding leads to our method of forward layered LDPC coupling described in section III. Another novelty in our paper is the definition of a degree distribution for the spatial coupling. Edges connecting two neighboring ensembles are not part of their protographs, these edges are defined by a separate degree distribution. Non-uniform chains of spatial coupling are introduced in section IV by gluing non-identical LDPC ensembles. Section V presents a space-varying coupled ensemble where the local ensemble and the local spatial coupling both depend on the spatial position. Conclusions and perspectives are drawn in section VI. In the sequel, the binary erasure channel (BEC) is the default transmission channel. All results are extendable to binary memoryless symmetric (BMS) channels.

II. Uniform spatial coupling

This section gives a very brief overview of uniform spatial coupling, i.e. the standard coupling as known in the literature. Consider a (d_b, d_c)-regular binary LDPC ensemble with M bit nodes of degree d_b and $\frac{d_b}{d_c}M$ check nodes of degree d_c. Edges connecting bit nodes and check nodes are given by a random matching between the $d_b \times M$ sockets of bit nodes and the $d_b \times M$ sockets of check nodes, see section 3.4 in [22]. It is assumed that M is infinitely large. This ensemble is referred to as the *uncoupled ensemble*. The design rate is $1 - \frac{d_b}{d_c}$ and $\frac{d_b}{d_c} = h^S$ is the Shannon threshold [10].

Building a coupled (d_b, d_c, L, w) ensemble works as follows. Define L spatial positions on a horizontal line. On each spatial position i, $i = 1 \ldots L$, place M bit nodes and $\frac{d_b}{d_c}M$ check nodes. Extra check nodes can be added at positions $i < 1$ and $i > L$ for the purpose of left and right termination of the coupled chain. The d_b edges of a bit node at position i are assumed to be uniformly and independently connected to check nodes in the range $[i, i+w-1]$. The d_c edges of a check node at position i are assumed to be uniformly and independently connected to bit nodes in the range $[i-w+1, i]$. The smoothing parameter w can be seen as the memory of LDPC convolutional codes built from the (d_b, d_c, L, w) coupled ensemble. Such coupling is called *uniform* for two reasons, connections are uniform in $[i, i+w-1]$ and the L uncoupled ensembles are all regular with identical degrees.

On BMS channels, two key results have been recently proven. The first one stated in Theorem 41 in [10] is

$$\lim_{w\to\infty} \lim_{L\to\infty} h^{BP}_{coupled} = h^{A}_{uncoupled}, \tag{1}$$

where h^{BP} is the threshold under Belief Propagation (BP) decoding and h^A is the area threshold ($h^A = h^{MAP}$ for the BEC). The second result stated in Lemma 29 in [10] is, for a given design rate,

$$\lim_{d_b\to\infty} h^{A}_{uncoupled} = h^{S}. \tag{2}$$

In the rest of this paper, we restrict our study to spatial coupling with a short memory $w = 2$.

III. Forward Layered LDPC Coupling

Layered coding is usually constructed via binning or superposition [21]. Another simple method to build layered codes for UEP is overlapping. The output of a first encoder is partially or fully fed to the input of a second encoder. The process is repeated for a bunch of L encoders. An example of overlapped layered coding is shown in Figure 1. Firstly, a direct sum of L regular $(3, 6)$ LDPC codes is constructed by copying the $(3, 6)$ parity-check matrix on the main diagonal. Then, each $(3, 6)$ parity-check matrix is connected to its next neighbor via a $(1, 2)$ sparse matrix, i.e. a random sparse binary matrix with weight one per column and weight two per row. The parity-check structure in Figure 1 is very similar to a coupled $(4, 8, L, w = 2)$ ensemble but it differs in how edges connect neighboring copies. More precisely, for a window length $w = 2$, forward layered coupling allows us to introduce a new non-uniformly coupled ensemble denoted by its parameters $(d_b, d_c; d_{bs}, d_{cs}; L)$. This new ensemble is an alternative to the uniform coupling ensemble $(d_b + d_{bs}, d_c + d_{cs}, L, w = 2)$.

Construction of the coupled $(d_b, d_c; d_{bs}, d_{cs}; L)$ ensemble: Define L spatial positions on a horizontal line. On each spatial position i, $i = 1 \ldots L$, place M bit nodes and $\frac{d_b}{d_c}M$ check nodes. Extra check nodes can be added at position $i > L$ for the purpose of right termination of the coupled chain. The d_b edges of a bit node at position i are assumed to be connected to check nodes in position i. The d_c edges of a check node at position i are assumed to be connected to bit nodes in position i. Spatial forward layered coupling is composed by extra spatial coupling edges. Each bit node at position i has extra d_{bs} edges connected to check nodes in position $i+1$. Each check node at position i has extra d_{cs} edges connected to bit nodes in position $i-1$. The (d_{bs}, d_{cs}) spatial coupling should satisfy the layered coding constraint $\frac{d_{bs}}{d_{cs}} = \frac{d_b}{d_c}$ translating the fact that both (d_b, d_c) and (d_{bs}, d_{cs}) matrices are of same size. The structure in Figure 1 corresponds to a $(3, 6; 1, 2; L)$ ensemble. Its compact graph representation (with protographs) is depicted in Figure 2.

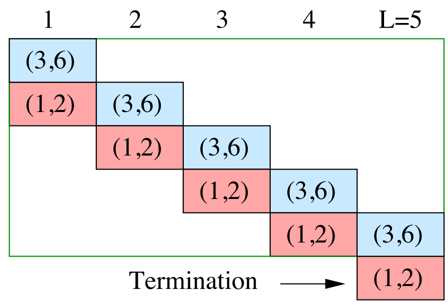

Figure 1. Parity-check matrix of overlapped L-layered coding built from $(3, 6)$-regular LDPC ensembles coupled via $(1, 2)$ sparse binary matrices.

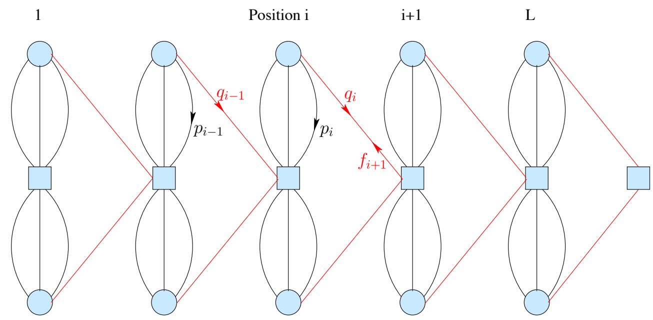

Figure 2. Tanner graph for $(3, 6)$-regular forward layered spatial coupling.

Without chain termination, the design rate for the $(d_b, d_c; d_{bs}, d_{cs}; L)$ ensemble is $1 - \frac{d_b}{d_c}$. The forward coupling structure has no left termination. A chain termination can be placed on the right of the coupled chain, i.e. an extra (d_{bs}, d_{cs}) local ensemble with its check nodes at $i = L + 1$. The rate loss due to termination is a factor of $1 - 1/L$. Other types of chain termination are possible for uniform and forward coupling, they will not be discussed in this paper.

Density evolution (DE) equations for a uniform spatially coupled (d_b, d_c, L, w) ensemble over the BEC(ϵ) are [9]

$$p_i = \epsilon \left(1 - \frac{1}{w}\sum_{j=0}^{w-1}\left(1 - \frac{1}{w}\sum_{k=0}^{w-1} p_{i+j-k}\right)^{d_c-1}\right)^{d_b-1} \tag{3}$$

for $i = 1 \ldots L$. For $w = 2$, right termination is incorporated by defining $p_{L+1} = 0$. DE fixed points equations for a spatially coupled $(d_b, d_c; d_{bs}, d_{cs}; L)$ ensemble require more messages because of the multiple edge types. Let p_i denote the message from bit node to check node in position i and q_i the ongoing message to check node in position $i + 1$.

Proposition 1: Let d_b, d_c, d_{bs}, and d_{cs} be positive integers satisfying $\frac{d_{bs}}{d_{cs}} = \frac{d_b}{d_c}$. DE equations for a forward layered spatially coupled $(d_b, d_c; d_{bs}, d_{cs}; L)$ ensemble are

$$p_i = \epsilon \cdot f_{i+1}^{d_{bs}} \cdot \left(1-(1-p_i)^{d_c-1}(1-q_{i-1})^{d_{cs}}\right)^{d_b-1},$$

$$q_i = \epsilon \cdot f_{i+1}^{d_{bs}-1} \cdot \left(1-(1-p_i)^{d_c-1}(1-q_{i-1})^{d_{cs}}\right)^{d_b},$$

$$f_{i+1} = \left(1-(1-p_{i+1})^{d_c}(1-q_i)^{d_{cs}-1}\right),$$

for $i = 1 \ldots L$, where f_{i+1} is a check-to-bit message propagating backward from position $i+1$ to position i.
Proof: From the compact graph representing the coupled chain in Figure 2, a reader who is familiar with modern coding theory [22] can quickly check that DE equations for the $(3, 6; 1, 2; L)$ ensemble are

$$p_i = \epsilon \cdot f_{i+1} \cdot \left(1-(1-p_i)^5(1-q_{i-1})^2\right)^2,$$

$$q_i = \epsilon \cdot \left(1-(1-p_i)^5(1-q_{i-1})^2\right)^3,$$

$$f_{i+1} = \left(1-(1-p_{i+1})^6(1-q_i)\right).$$

Then, it is straightforward to generalize to any integer parameters defining the coupled ensemble. The equal ratios between the degrees of the local ensemble and the degrees of the spatial coupling force bit nodes and check nodes to have equal number of sockets for connecting edges. *QED*.

Using (3) and Proposition 1, belief propagation thresholds are computed on the BEC for different rate-$1/2$ ensembles. Thresholds are listed in Table I for $d_{bs} = 1$ and $d_{cs} = 2$. A uniform spatially coupled $(d_b, d_c, L, w = 2)$ ensemble (column 4) is compared to a forward layered $(d_b-1, d_c-2; 1, 2; L)$ ensemble (column 5). For both ensembles, right chain termination is made by $p_{L+1} = 0$. Any chain length L sufficiently large yields same threshold results, typical values are $L = 60$ or $L = 100$. Table I shows that forward layered LDPC coupling outperforms the standard uniform spatial coupling.

Ensemble	$h^{BP}_{uncoupled}$	$h^{MAP}_{uncoupled}$	$h^{BP}_{uniform}$	$h^{BP}_{forward}$
(3,6)	0.42944	0.48815	0.48808	0.48815
(4,8)	0.38345	0.49774	0.49442	0.49741
(5,10)	0.34155	0.49948	0.48268	0.49811
(6,12)	0.30746	0.49988	0.46031	0.49667

Table I
BEC THRESHOLDS OF UNIFORM SPATIAL COUPLING VERSUS FORWARD LAYERED COUPLING, $w = 2$, $d_{bs} = 1$, AND $d_{cs} = 2$.

Another illustration of the forward spatial coupling performance is given in Figure 3. BEC thresholds are plotted versus the spatial position i/L, for $i = 1 \ldots L$, for the $(3, 6; 1, 2; L)$ ensemble. The coupled ensemble with termination largely overpasses the thresholds of the uncoupled $(3, 6)$ and $(4, 8)$ LDPC. In fact, it saturates at 0.49741 very close to the $(4, 8)$-LDPC MAP threshold, see the $(4, 8)$ row in Table I. The small value $L = 10$ is used for the purpose of illustration, its rate loss makes it unacceptable in practice. Without chain termination, forward layered coupling improves the threshold but stays blocked at a relatively small value of 0.44695. Uniform coupling is capable of saturating the threshold without chain termination but unfortunately, in the case of $w = 2$, it may need up to 6 spatial positions to push up the threshold from right to left starting at $i = L$. This loss in local thresholds cannot be tolerated for finite L. Hence, we compare both coupled ensembles under right termination for a typical finite value of chain length L.

Results in Figure 3 are found for 10000 decoding iterations. In order to make the threshold flat with respect to the spatial position i/L, the number of decoding iterations increases with L. Furthermore, the $w = 2$ forward spatial coupling lets a (d_b, d_c) ensemble at position i strengthen its neighbor at position $i-1$, i.e. bit nodes protection propagates from right to left due to the structure of the (d_{bs}, d_{cs}) spatial coupling. Indeed, coupling edges leave a check node at position i to reach a bit node at position $i-1$.

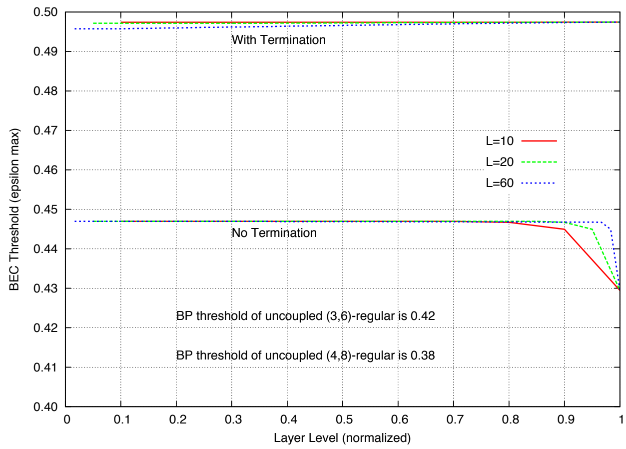

Figure 3. Performance of the forward coupled $(3, 6; 1, 2; L)$ ensemble.

In the next section, LDPC ensembles with different parameters d_b and d_c are coupled. The local (d_b, d_c) ensemble is kept regular (i.e. d_b and d_c are integer), but the (d_{bs}, d_{cs}) spatial coupling is not necessarily regular. The objective is to create a stair of thresholds for unequal error protection.

IV. NON-UNIFORM CHAIN OF SPATIAL COUPLING

Following the same arguments and notations as in previous sections, let us build a non-uniform chain by gluing non-identical LDPC ensembles.

Definition 1: Consider two coupled ensembles $(d_b, d_c; d_{bs}, d_{cs}; L_1)$ and $(d'_b, d'_c; d'_{bs}, d'_{cs}; L_2)$ corresponding to two chains of length L_1 and L_2 respectively. The new ensemble defined by gluing the two chains is written as

$$(d_b, d_c; d_{bs}, d_{cs}; L_1) \oplus (d'_b, d'_c; d'_{bs}, d'_{cs}; L_2)$$

where the new chain has length $L = L_1 + L_2$.

In general, a chain of length $L = \aleph L_1$ is built by gluing $\aleph$ sub-chains. The new coupled ensemble can be written as

$$\bigoplus_{j=1}^{\aleph} (d_b(j), d_c(j); d_{bs}(j), d_{cs}(j); L_1). \tag{4}$$

In this section, $d_b(j)$ and $d_c(j)$ are integers but not necessarily $d_{bs}(j)$ and $d_{cs}(j)$, i.e. the spatial coupling is irregular. Long and complex degree distributions are unnecessary. We will define and use the simplest irregular distributions.

Definition 2: $\Psi(\alpha, d)$ is a parametric polynomial that includes two monomials as follows

$$\Psi(\alpha, d) = \alpha x^d + (1-\alpha) x^{d+1}.$$

Let $d_{bs}(j)$ and $d_{cs}(j)$ be two positive real numbers. The real $d_{bs}(j)$ is the average degree of bit nodes at the left of a spatial coupling connecting two protographs of type $(d_b(j), d_c(j))$ inside sub-chain j. Similarly, $d_{cs}(j)$ is the average degree of check nodes at the right of a spatial coupling. Instead of showing a sub-chain section, Figure 4 displays a general section in the total chain between spatial positions i and $i+1$. If position i belongs to sub-chain j then, for example, $d_{b,i} = d_b(j)$ and $d_{bs,i} = d_{bs}(j)$. From a node perspective, the left and right degree distributions of the spatial coupling are taken to be

$$\begin{aligned} \mathring{\lambda}_s(x) &= \Psi(\lfloor d_{bs} \rfloor + 1 - d_{bs}, \lfloor d_{bs} \rfloor), \\ \mathring{\rho}_s(x) &= \Psi(\lfloor d_{cs} \rfloor + 1 - d_{cs}, \lfloor d_{cs} \rfloor). \end{aligned} \tag{5}$$

In multiple-edge-type LDPC structures, such as [23] [24], both node and edge perspectives are required in DE equations. From an edge perspective, left and right degree distributions associated to (5) are

$$\begin{aligned} \lambda_s(x) &= \Psi(\tfrac{\lfloor d_{bs} \rfloor}{d_{bs}}(\lfloor d_{bs} \rfloor + 1 - d_{bs}), \lfloor d_{bs} \rfloor - 1), \\ \rho_s(x) &= \Psi(\tfrac{\lfloor d_{cs} \rfloor}{d_{cs}}(\lfloor d_{cs} \rfloor + 1 - d_{cs}), \lfloor d_{cs} \rfloor - 1). \end{aligned} \tag{6}$$

From (5) or (6), the reader can easily check that given distributions have average degrees d_{bs} and d_{cs}. The non-uniform chain of length $L = \aleph L_1$ comprises $\aleph$ pairs $(d_{bs}(j), d_{cs}(j))$, for $j = 1 \ldots \aleph$.

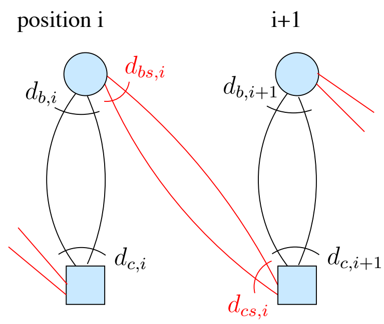

Figure 4. Protographs with $(d_{bs,i}, d_{cs,i})$ spatial coupling.

Density evolution fixed-point equations of Proposition 1 are now updated by incorporating polynomials from (5) and (6). Likewise polynomials used in DE equations for root-LDPC ensembles [23] [24], edge perspective is applied to messages passing through edges located in front of a node generating the DE message. *Node perspective* is applied to messages carried by edges located behind the node. In the chain defined by (4), DE equations at spatial position i are given by the following proposition.

Proposition 2: For $i = 1 \ldots L$, we have

$$p_i = \epsilon \cdot \mathring{\lambda}_s(f_{i+1}) \cdot \left(1 - (1-p_i)^{d_c - 1} \cdot \mathring{\rho}_s(1 - q_{i-1})\right)^{d_b - 1},$$

$$q_i = \epsilon \cdot \lambda_s(f_{i+1}) \cdot \left(1 - (1-p_i)^{d_c - 1} \cdot \mathring{\rho}_s(1 - q_{i-1})\right)^{d_b},$$

$$f_{i+1} = 1 - (1 - p_{i+1})^{d_c} \cdot \rho_s(1 - q_i),$$

where $d_b = d_{b,i} \in \mathbb{N}$, $d_c = d_{c,i} \in \mathbb{N}$, $d_{bs} = d_{bs,i} \in \mathbb{R}^+$, and $d_{cs} = d_{cs,i} \in \mathbb{R}^+$ with the following layered coding constraint $\frac{d_{bs,i}}{d_{cs,i}} = \frac{d_{b,i+1}}{d_{c,i+1}}$, i.e. the degree ratio for a spatial coupling at position i is imposed by the protograph at position $i+1$.

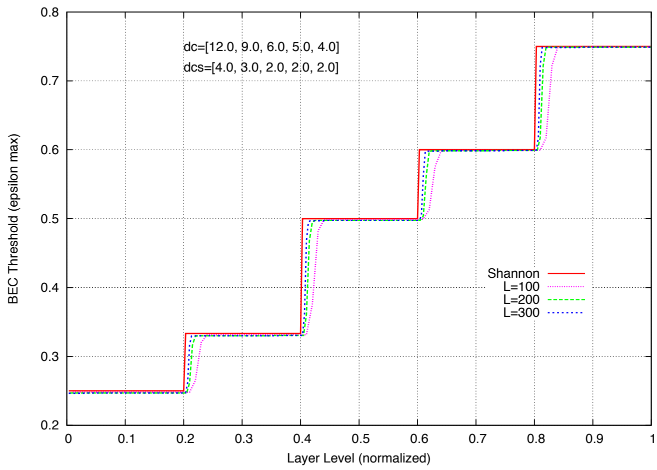

Figure 5. BEC thresholds for a non-uniform chain of forward spatial coupling with $\aleph = 5$ sub-chains each of length $L_1 = L/5$.

A stair for unequal error protection is plotted on Figure 5. The five uncoupled LDPC ensembles are, from left to right, $(3, 12)$, $(3, 9)$, $(3, 6)$, $(3, 5)$, and $(3, 4)$. The corresponding spatial coupling is $(1, 4)$, $(1, 3)$, $(1, 2)$, $(1.2, 2)$, and $(1.5, 2)$. The threshold stair is saturating up on Shannon stair defined by the five local Shannon thresholds.

V. Spatially-variant spatial coupling

Let us consider the highest number of sub-chains inside a non-uniform chain. This extremal situation corresponds to $\aleph = L$. The local LDPC ensemble has real average degrees $d_{b,i}$ and $d_{c,i}$ varying with respect to i, for $i = 1 \ldots L$. The irregularity of left and right degree distributions is taken into account by $\mathring{\lambda}(x)$, $\lambda(x)$, $\mathring{\rho}(x)$, and $\rho(x)$. These polynomials are defined in a similar fashion as in (5) and (6). For example, we have $\overset{\circ}{\lambda}(x) = \Psi(\lfloor d_b \rfloor + 1 - d_b, \lfloor d_b \rfloor)$. The spatially-variant chain becomes

$$\bigoplus_{i=1}^{L} (d_{b,i}, d_{c,i}; d_{bs,i}, d_{cs,i}; 1). \quad (7)$$

Proposition 3: Let $d_{b,i} \in \mathbb{R}^+$, $d_{c,i} \in \mathbb{R}$, $d_{bs,i} \in \mathbb{R}$, $d_{cs,i} \in \mathbb{R}$, satisfy $\frac{d_{bs,i}}{d_{cs,i}} = \frac{d_{b,i+1}}{d_{c,i+1}}$, for $i = 1 \ldots L$. DE equations for this spatially-variant ensemble are

$$p_i = \epsilon \cdot \overset{\circ}{\lambda}_{s,i}(f_{i+1}) \cdot \lambda_i \left(1 - \rho_i(1-p_i) \cdot \overset{\circ}{\rho}_{s,i-1}(1-q_{i-1})\right),$$

$$q_i = \epsilon \cdot \lambda_{s,i}(f_{i+1}) \cdot \overset{\circ}{\lambda}_i \left(1 - \rho_i(1-p_i) \cdot \overset{\circ}{\rho}_{s,i-1}(1-q_{i-1})\right),$$

$$f_{i+1} = 1 - \overset{\circ}{\rho}_{i+1}(1-p_{i+1})\rho_{s,i}(1-q_i).$$

Threshold results for a spatially-variant chain are plotted in Figure 6 for the binary erasure channel and $L = 200$.

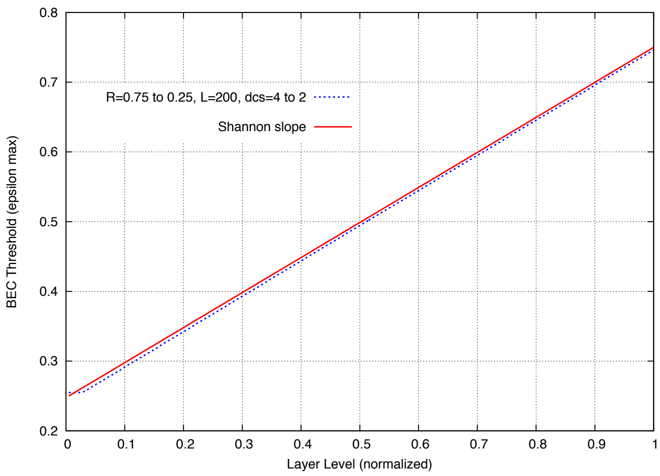

Figure 6. BEC thresholds for a non-uniform chain of forward spatial coupling with $\aleph = L$ local ensembles varying with the spatial position. Parameters are $L = 200$, $d_b = 3$, $d_c = 12$ to 4, $Rate = 0.75$ to 0.25.

VI. Conclusions

A new method, called forward layered coupling, for spatial coupling of low-density parity-check ensembles was proposed. The method is inspired from overlapped layered coding and has the shortest possible memory, i.e. $w = 2$. Edges of local ensembles and those defining the spatial coupling are separately built. Thresholds for regular coupled chains were given in Table I. The new method also allows the construction of non-uniform coupling chains with near-Shannon spatially-varying thresholds under iterative decoding. Numerical results with stair-like and slope-like thresholds were presented in Figures 5 and 6. Our future work should include potential applications of forward layered coupling in different areas in communications and coding.

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