A Poltyrev Outage Limit for Lattices

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***Abstract*—The non-ergodic fading channel is a useful model for various wireless communication channels in both indoor and outdoor environments. In this model, a codeword is divided into multiple blocks such that fading is constant within a block and independent across blocks. Building on Poltyrev's work on infinite lattice constellations for the Gaussian channel, we derive a Poltyrev outage limit for lattices in presence of block fading. We prove that the diversity order of this Poltyrev outage limit is equal to the number of degrees of freedom in the channel. An important application is lattice decoding of low-density lattice codes. Under block fading, it is well known that both sphere decoding and iterative decoding of low-density lattices are dramatically complex. The newly defined Poltyrev outage limit is used to declare an outage error without decoding, which drastically improves the decoding runtime.**

I. Introduction

Coded modulations are nowadays an integral part of almost all communication systems. Channel coding and linear modulation signal sets can be combined together in many different ways [1]. Lattices in the real Euclidean space are a special case of coded modulations. Lattices are infinite constellations of points with a group structure [2]. With lattice points perturbed by additive white Gaussian noise, Poltyrev showed that there exists a lattice that can be correctly decoded if and only if the noise variance is less than a threshold [5]. This threshold will be called *Poltyrev threshold* in the sequel and corresponds to data transmission at infinite rate. Its counterpart for finite-rate data transmission is Shannon capacity [4].

A finite lattice constellation is a finite subset of a lattice defined by the intersection of the lattice with a shaping region centered at the origin. The shaping region itself can be defined by the Voronoi region of another lattice or a sub-lattice [3]. It was shown that the capacity of a Gaussian channel can be achieved via a finite lattice constellation [9]. In order to achieve capacity, a lattice must have two properties: Gaussian goodness and covering goodness. The Gaussian goodness means that the lattice can achieve Poltyrev threshold. The covering goodness is equivalent to a shaping with the best second order moment. Hence, before attaining Shannon capacity with a finite constellation, an infinite lattice constellation should attain Poltyrev threshold. This is a main contribution from the infinite-rate case to the finite-rate case. In this paper, we only focus on infinite lattice constellations.

Poltyrev's work does not restrict the dimension of the infinite constellation that achieves a vanishing error probability. Infinite constellations with a finite dimension were analyzed by Ingber *et al.* in [6] over the Gaussian channel. Authors in [7] studied the diversity-multiplexing tradeoff of infinite constellations over multiple-antenna fading channels. In this paper, we consider transmission using lattices over a single-antenna block-fading (BF) channel. Using Poltyrev threshold, we derive a *Poltyrev outage limit* for lattices over block-fading channels. We prove that Poltyrev outage limit has diversity L for a channel with L independent block fadings, i.e. Poltyrev outage limit has full diversity. One of the most important applications of Poltyrev outage limit is to detect inadmissible channel states. Inadmissible fadings are deep fadings that cause an outage event. We utilize Poltyrev outage limit to render feasible the decoding of a low-density lattice in presence of block fading. The rest of the paper is structured as follows. Background information and notation are given in Section II. Poltyrev outage limit for lattices over block fading channels is derived in Section III. Section IV gives the proof for the diversity order of the Poltyrev outage limit. Simulation results for sphere decoding of low-density lattice codes are provided in Section V before the conclusion in Section VI.

II. Background and Notation

A. Lattices

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{Z}$ its ring of integers, and $\mathbb{Q}$ the field of fractions of $\mathbb{Z}$. A *lattice* $\Lambda \subset \mathbb{R}^n$, also called a point lattice, is a free $\mathbb{Z}$-module of rank n in $\mathbb{R}^n$. An element belonging to Λ is called a *point* or equivalently a vector. Any point $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T \in \Lambda$ can be written as an integer linear combination of n points $\boldsymbol{x} = \sum_{i=1}^{n} z_i \boldsymbol{v}_i$, where $\{\boldsymbol{v}_i\}$ is a $\mathbb{Z}$-basis of Λ, $v_{i,j} \in \mathbb{R}$, and $z_i \in \mathbb{Z}$. The $n \times n$ matrix $\mathbf{G}$ built from a basis is a *generator matrix* for Λ. In column convention, let $\mathbf{G} = [v_{i,j}]$, then a lattice point is written as $\boldsymbol{x} = \mathbf{G}\boldsymbol{z}$, where $\boldsymbol{z} \in \mathbb{Z}^n$. The *fundamental volume* of the lattice Λ is given by $|\det(\mathbf{G})|$. For more information on lattices, we refer the reader to [2].

Poltyrev threshold can be stated as follows: There exists a lattice Λ of high enough dimension n for which the transmission error probability over an additive white Gaussian noise (AWGN) channel can be reduced to an arbitrary low level if and only if $\sigma^2 < \sigma^2_{max}$ [5],[8] where σ^2 is the noise variance per dimension and Poltyrev threshold σ^2_{max} is given by

$$\sigma^2_{max} = \frac{|\det(\mathbf{G})|^{\frac{2}{n}}}{2\pi e}. \tag{1}$$

An integer-check matrix of a lattice is the inverse of a generator matrix, $\mathbf{H} = [\mathbf{H}_{i,j}] = \mathbf{G}^{-1}$. The n integer-check equations for a lattice point $\boldsymbol{x}$ are $\sum_j \mathbf{H}_{i,j}\ x_j \in \mathbb{Z}$, for $i = 1 \ldots n$. *Low-density lattice codes* (LDLCs) are a special class of lattices proposed by Sommer *et al.* in [8]. An LDLC is defined by a sparse integer-check matrix in order to allow for iterative decoding in high dimensions. Other families of lattices built from sparse codes have been recently proposed, such as lattices from Construction A with non-binary low-density parity-check codes [10], referred to as LDA lattices. More recently, a powerful family of GLD lattices for the AWGN channel has been defined by the intersection of repeated and interleaved lattices [11]. Standard LDLC, LDA, and GLD lattices exhibit no diversity and should be modified as in (34) in presence of block fading.

B. Non-Ergodic Fading Channel Model

We assume coherent detection with perfect channel state information at the receiver side only. The fading channel is flat, i.e., there are no multiple paths [1]. Fading coefficients are real non-negative with a Rayleigh distribution. If α denotes a fading coefficient, then $p(\alpha) = 2\alpha \exp(-\alpha^2)$ or equivalently $p(\alpha^2) = \exp(-\alpha^2)$, for $0 \leq \alpha < +\infty$. It is worth noting that results in this paper do not rely on this particular distribution of fading, they are still valid for most usual fading distributions, e.g. the Nakagami distribution of order m.

Let $\boldsymbol{x}$ be a lattice point. Consider the non-ergodic fading where fading coefficients take only L values within a lattice point, $2 \leq L < n$. The non-ergodic BF channel with diversity L has the following mathematical model:

$$y_i = \alpha_j x_i + \eta_i, \quad j = \left\lceil \frac{i}{n/L} \right\rceil, \quad i = 1, 2, \ldots n, \qquad (2)$$

where α_j are independent and identically distributed (i.i.d.) Rayleigh distributed fading coefficients and $\eta_i \sim \mathcal{N}(0, \sigma^2)$.

Let γ be the signal-to-noise ratio (SNR) for an infinite lattice constellation,

$$\gamma = \frac{|\det(\mathbf{G})|^{\frac{2}{n}}}{\sigma^2}. \qquad (3)$$

Assuming maximum-likelihood (ML) decoding at the receiver side, let $P_e(\Lambda)$ be the point error probability. On the BF channel defined by (2), the diversity order of $P_e(\Lambda)$ is defined by its slope at high SNR [1]

$$\lim_{\gamma \to \infty} \frac{-\log(P_e(\Lambda))}{\log(\gamma)}. \qquad (4)$$

If f and g have the same diversity order then we denote this by $f(\gamma) \doteq g(\gamma)$, i.e. $\frac{-\log(f)}{\log(\gamma)} = \frac{-\log(g)}{\log(\gamma)}$ for $\gamma \to \infty$. Similarly we introduce the notation with inequalities $\dot{\leq}$ and $\dot{\geq}$.

Definition 1. *Consider a BF channel with L independent fading coefficients per lattice point. Λ is a full-diversity lattice under ML decoding if $P_e(\Lambda) \doteq \frac{1}{\gamma^L}$.*

III. Poltyrev Outage Limit For Infinite Lattice Constellations

In his work on full-diversity low-density lattice codes, the author of [13] could not get numerical results for dimensions above 16. The sphere decoding algorithm [14] stuck on some lattice points enduring moderately deep or very deep fading. Also, the LDLC team of [7][8] could not get numerical results for iterative decoding on the BF channel at high dimensions. Furthermore, the result of a double-diversity infinite lattice constellation in dimension 8 presented in [13] was compared to a lattice with no diversity (i.e. diversity order of 1). No attempt has been made in the literature to derive a limit for the BF channel equivalent to that of Poltyrev for the Gaussian channel.

Let $\boldsymbol{\alpha} = \mathrm{diag}\,(\alpha_1, \,.\,.\,, \alpha_1, \alpha_2, \,.\,.\,, \alpha_2, \ldots, \alpha_L, \,.\,.\,, \alpha_L)$ be the $n \times n$ diagonal matrix including the L fading coefficients, each repeated n/L times as defined by the model in (2). Given the lattice generator matrix $\mathbf{G}$, after going through the BF channel, the new lattice added to the AWG noise has the generator matrix $\mathbf{G}_{\text{new}} = \boldsymbol{\alpha}\mathbf{G}$. The fundamental volume becomes

$$|\det(\mathbf{G}_{\text{new}})| = |\det(\mathbf{G})| \times \prod_{l=1}^{L} \alpha_l^{n/L}. \qquad (5)$$

For a fixed instantaneous fading $\boldsymbol{\alpha}$, after combining (1) and (5), Poltyrev threshold becomes

$$\sigma^2_{max}(\boldsymbol{\alpha}) = \frac{\prod_{l=1}^{L} \alpha_l^{2/L} |\det(\mathbf{G})|^{\frac{2}{n}}}{2\pi e}. \qquad (6)$$

Decoding of the infinite lattice constellation is possible with a vanishing error probability if $\sigma^2 < \sigma^2_{max}(\boldsymbol{\alpha})$ [5]. Hence, for variable fading, an outage event occurs whenever $\sigma^2 > \sigma^2_{max}(\boldsymbol{\alpha})$. The Poltyrev outage limit $P_{\text{out}}(\gamma)$ is then defined by the following probability

$$\begin{aligned} P_{\text{out}}(\gamma) &= P\left(\sigma^2 > \frac{\prod_{l=1}^{L} \alpha_l^{2/L} |\det(\mathbf{G})|^{\frac{2}{n}}}{2\pi e}\right) \\ &= P\left(\prod_{l=1}^{L} \alpha_l^2 < \frac{(2\pi e)^L}{\gamma^L}\right). \end{aligned} \qquad (7)$$

$P_{out}(\gamma)$ does not admit a closed-form expression but it can be numerically estimated via the Monte Carlo method. The point error rate after lattice decoding, for a given lattice over a BF channel, can be compared to $P_{out}(\gamma)$ to validate the diversity order and the gap in signal-to-noise ratio. But most importantly, the equality $\prod_{l=1}^{L} \alpha_l^2 = \frac{(2\pi e)^L}{\gamma^L}$ defines a boundary in the fading space below which outage events occur. This boundary shall be called Poltyrev outage boundary. In the next section we prove that $P_{\text{out}}(\gamma) \doteq \frac{1}{\gamma^L}$.

IV. Diversity Order of Poltyrev Outage Limit

It is well known from Maximum Ratio Combining techniques on fading channels [1] that $P(\sum_{l=1}^{L} \alpha_l^2 < \frac{1}{\gamma})$ has diversity L. Is it true for $\prod_l \alpha_l^2$? Fortunately, the expression of Poltyrev outage limit in (7) compares the product of squared fadings to $\frac{1}{\gamma^L}$, not to $\frac{1}{\gamma}$. Roughly speaking, $P_{\text{out}}(\gamma)$ behaves like $[P(\alpha_1^2 < \frac{1}{\gamma})]^L$ leading to diversity L. Let us make the exact proof.

The proof of $P_{\text{out}}(\gamma) \doteq \frac{1}{\gamma^L}$ is made by induction; first for the special case of $L = 2$ and later for an arbitrary value of L. The constant term $2\pi e$ can be embedded into γ, so we have

$$P_{out}(\gamma) \doteq P\left(\gamma^L \prod_{l=1}^{L} \alpha_l^2 < 1\right). \qquad (8)$$

The equality in diversity is reached after proving a lower bound and an upper bound for $P_{out}(\gamma)$. In other words, $P_{out}(\gamma) \dot{\geq} \frac{1}{\gamma^L}$ and $P_{out}(\gamma) \dot{\leq} \frac{1}{\gamma^L}$ is equivalent to $P_{\text{out}}(\gamma) \doteq \frac{1}{\gamma^L}$. The reader should be aware that a lower bound on the error probability yields an upper bound on diversity, and vice versa.

Lemma 1. *Consider a BF channel with diversity $L = 2$. The Poltyrev outage limit defined by (7) satisfies $P_{out}(\gamma) \doteq \frac{1}{\gamma^2}$.*

Proof: We now prove $P_{out}(\gamma) \doteq P\left(\alpha_1^2\alpha_2^2\,\gamma^2 < 1\right) \doteq \frac{1}{\gamma^2}$. Recall that a Rayleigh distributed random variable α_i satisfies, for any $k > 0$,

$$P\left(\alpha_i^2 < \frac{1}{\gamma^k}\right) \doteq \frac{1}{\gamma^k} \ . \tag{9}$$

1) Lower Bound:

$$P\left(\alpha_1^2\alpha_2^2\,\gamma^2 < 1\right) = P\left(\alpha_1^2\alpha_2^2\,\gamma^2 < 1 \mid \alpha_2^2 < 1\right) P\left(\alpha_2^2 < 1\right) + P\left(\alpha_1^2\alpha_2^2\,\gamma^2 < 1 \mid \alpha_2^2 > 1\right) P\left(\alpha_2^2 > 1\right) \ .$$

We lower bound by the first term to get

$$\begin{aligned} P\left(\alpha_1^2\alpha_2^2\,\gamma^2 < 1\right) &\geq P\left(\alpha_1^2\alpha_2^2\,\gamma^2 < 1 \mid \alpha_2^2 < 1\right) P\left(\alpha_2^2 < 1\right) \\ &\geq P\left(\alpha_1^2\,\gamma^2 < 1\right) P\left(\alpha_2^2 < 1\right) \\ &\doteq P\left(\alpha_1^2\,\gamma^2 < 1\right), \end{aligned}$$

which gives

$$P\left(\alpha_1^2\alpha_2^2\,\gamma^2 < 1\right) \dot{\geq} \frac{1}{\gamma^2} \ . \tag{10}$$

2) Upper Bound: To derive an upper bound on $P\left(\alpha_1^2\alpha_2^2\,\gamma^2 < 1\right)$, we plot the boundary $\alpha_1^2\alpha_2^2 = \frac{1}{\gamma^2}$ which is a hyperbola as shown in blue in Fig. 1 (to be taken for $L = 2$). We partition the total area under the outage boundary into three regions: a) $\mathcal{R}$ is the area where $\alpha_1^2 < \frac{1}{\gamma}$ and $\alpha_2^2 < \frac{1}{\gamma}$; b) $\mathcal{T}_1$ is the area where $\alpha_1^2 < \frac{1}{\gamma}$ and $\alpha_2^2 > \frac{1}{\gamma}$ but $\alpha_1^2\alpha_2^2 < \frac{1}{\gamma^2}$; and c) $\mathcal{T}_2$ is the area where $\alpha_1^2 > \frac{1}{\gamma}$ and $\alpha_2^2 < \frac{1}{\gamma}$ but $\alpha_1^2\alpha_2^2 < \frac{1}{\gamma^2}$. The areas $\mathcal{T}_1$ and $\mathcal{T}_2$ are equal (for $L = 2$). So we can write

$$P\left(\alpha_1^2\alpha_2^2\,\gamma^2 < 1\right) = P(\mathcal{R}) + 2P(\mathcal{T}_2) \ . \tag{11}$$

Then we have

$$\begin{aligned} P(\mathcal{R}) &= P\left(\alpha_1^2 \in \left[0, \frac{1}{\gamma}\right] \text{ and } \alpha_2^2 \in \left[0, \frac{1}{\gamma}\right]\right) \\ &= \left(P\left(\alpha_1^2 \in \left[0, \frac{1}{\gamma}\right]\right)\right)^2 = \left(P\left(\alpha_1^2 \leq \frac{1}{\gamma}\right)\right)^2 \doteq \frac{1}{\gamma^2}. \end{aligned} \tag{12}$$

We now introduce $\phi(x) = 1 - e^{-x}$ which is required in the sequel. For $x \geq 0$, it can be shown that

$$\phi(x) = 1 - e^{-x} \leq \min(1, x) \ . \tag{13}$$

Now denote $X = \alpha_1^2$ and $Y = \alpha_2^2$. Then

$$P(\mathcal{T}_2) = \int_{\mathcal{T}_2} p_{X,Y}(x, y) = \int_{x=\frac{1}{\gamma}}^{\infty} \int_{y=0}^{\frac{1}{x\gamma^2}} e^{-x} e^{-y} dx\, dy, \tag{14}$$

after integrating over y, we get

$$P(\mathcal{T}_2) = \int_{x=\frac{1}{\gamma}}^{\infty} \left(1 - e^{-\frac{1}{x\gamma^2}}\right) e^{-x}\, dx. \tag{15}$$

The previous integral has $\frac{1}{x\gamma^2} \leq \frac{1}{\gamma} \leq 1$ at high SNR. We get

$$\phi\left(\frac{1}{x\gamma^2}\right) \leq \min\left(1, \frac{1}{x\gamma^2}\right) = \frac{1}{x\gamma^2} \ .$$

With this result, (15) can be upperbounded as

$$P(\mathcal{T}_2) \leq \int_{x=\frac{1}{\gamma}}^{\infty} \frac{1}{x\gamma^2} e^{-x}\, dx = \frac{\Delta}{\gamma^2} \ , \tag{16}$$

where Δ is defined as

$$\Delta = \int_{x=\frac{1}{\gamma}}^{\infty} \frac{e^{-x}}{x}\, dx. \tag{17}$$

Solving (17) using integration by parts yields

$$\Delta = e^{-\frac{1}{\gamma}} \ln(\gamma) + \int_{\frac{1}{\gamma}}^{\infty} \ln(x) e^{-x} dx. \tag{18}$$

With $e^{-\frac{1}{\gamma}} \leq 1$ and $\ln(x) \leq x$, (18) can be upperbounded as

$$\Delta \leq \ln(\gamma) + \frac{1}{\gamma} e^{-\frac{1}{\gamma}} + e^{-\frac{1}{\gamma}} \tag{19}$$

Substituting the upper bound on Δ in (16),

$$P(\mathcal{T}_2) \leq \frac{\ln(\gamma) + \frac{1}{\gamma} e^{-\frac{1}{\gamma}} + e^{-\frac{1}{\gamma}}}{\gamma^2} \ \Rightarrow\ P(\mathcal{T}_2) \dot{\leq} \frac{1}{\gamma^2} \tag{20}$$

Using (12) and (20) in (11), we have

$$P\left(\alpha_1^2\alpha_2^2\,\gamma^2 < 1\right) \ \dot{\leq} \ \frac{1}{\gamma^2} \tag{21}$$

which is the desired upper bound.
It can be concluded from the upper bound and lower bound as given in (10) and (21) that

$$P\left(\alpha_1^2\alpha_2^2\,\gamma^2 < 1\right) \doteq \frac{1}{\gamma^2} \ .$$

■

Now the previous lemma is generalized by induction to an arbitrary value of diversity order $L \geq 2$.

Theorem 1. *Consider a BF channel with diversity $L \geq 2$. The Poltyrev outage limit defined by (7) satisfies $P_{out}(\gamma) \doteq \frac{1}{\gamma^L}$.*

Proof: Let us assume that the theorem statement is true for $L - 1$, i.e.

$$P\left(\prod_{i=1}^{L-1} \alpha_i^2\ \gamma^{(L-1)} < 1\right) \doteq \frac{1}{\gamma^{(L-1)}} \ . \tag{22}$$

Now, let us prove it for a diversity order L. As for Lemma 1, we derive upper and lower bounds for $P_{out}(\gamma)$.

1) Lower Bound: In a way similar to the proof of the lower bound in Lemma 1, the term with $\alpha_L^2\gamma > 1$ is dropped, so we have

$$\begin{aligned} P_{out}(\gamma) &\doteq P\left(\prod_{i=1}^{L-1} \alpha_i^2\ \gamma^{(L-1)}\ \alpha_L^2\gamma < 1\right) \\ &\geq P\left(\prod_{i=1}^{L-1} \alpha_i^2\ \gamma^{(L-1)} < 1\right) P\left(\alpha_L^2\gamma < 1\right) \\ &\dot{\geq} \frac{1}{\gamma^{(L-1)}\gamma} \ = \ \frac{1}{\gamma^L} \ , \end{aligned}$$

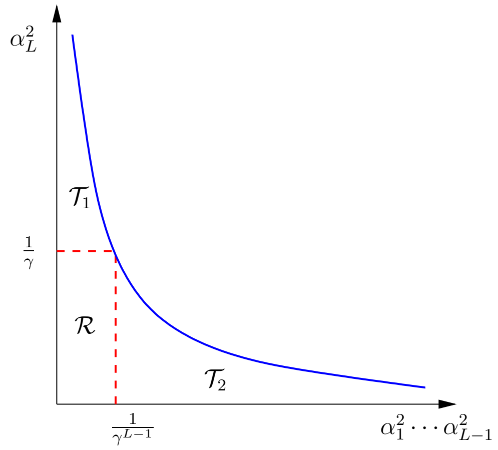

Fig. 1: Boundary in the fading plane defined by a constant product of squared fadings at a given SNR, $\alpha_1^2\alpha_2^2\cdots\alpha_L^2 = \frac{1}{\gamma^L}$. The area under the Poltyrev outage boundary is partitioned into three regions $\mathcal{R}$, $\mathcal{T}_1$, and $\mathcal{T}_2$.

which gives the required lower bound with diversity L. The upper bound relies on the partitioning of the area under the outage boundary.

2) Upper Bound: We partition the area under the outage boundary into three regions: a) $\mathcal{R}$ is the area where $\prod_{i=1}^{L-1}\alpha_i^2 < \frac{1}{\gamma^{L-1}}$ and $\alpha_L^2 < \frac{1}{\gamma}$; b) $\mathcal{T}_1$ is the area where $\prod_{i=1}^{L-1}\alpha_i^2 < \frac{1}{\gamma^{L-1}}$ and $\alpha_L^2 > \frac{1}{\gamma}$ but $\prod_{i=1}^{L}\alpha_i^2 < \frac{1}{\gamma^L}$; and c) $\mathcal{T}_2$ is the area where $\prod_{i=1}^{L-1}\alpha_i^2 > \frac{1}{\gamma^{L-1}}$ and $\alpha_L^2 < \frac{1}{\gamma}$ but $\prod_{i=1}^{L}\alpha_i^2 < \frac{1}{\gamma^L}$. The areas $\mathcal{T}_1$ and $\mathcal{T}_2$ are not equal for $L > 2$. With this, we get

$$P_{out}(\gamma) \doteq P(\mathcal{R}) + P(\mathcal{T}_1) + P(\mathcal{T}_2). \tag{23}$$

$$\begin{aligned} P(\mathcal{R}) &= P\left(\alpha_L^2 \in \left[0, \frac{1}{\gamma}\right] \text{ and } \alpha_1^2\cdots\alpha_{L-1}^2 \in \left[0, \frac{1}{\gamma^{(L-1)}}\right]\right) \\ &= P\left(\alpha_L^2 \le \frac{1}{\gamma}\right) P\left(\alpha_1^2\cdots\alpha_{L-1}^2 \le \frac{1}{\gamma^{(L-1)}}\right) \\ &\doteq \frac{1}{\gamma^L}. \end{aligned} \tag{24}$$

The last equality is derived from (9) and (22). For the calculation of $P(\mathcal{T}_1)$ and $P(\mathcal{T}_2)$, let $X = \prod_{i=1}^{(L-1)}\alpha_i^2$ and let $Y = \alpha_L^2$.

Calculation of $P(\mathcal{T}_1)$:

$$P(\mathcal{T}_1) = \int_{y=\frac{1}{\gamma}}^{\infty} e^{-y}\, dy \int_{x=0}^{\frac{1}{y\gamma^L}} p_X(x)\, dx \tag{25}$$

From (22) we get,

$$\int_{x=0}^{\frac{1}{y\gamma^L}} p_X(x)\, dx = P\left(\prod_{i=1}^{L-1}\alpha_i^2 < \frac{1}{y\gamma^L}\right) \doteq \frac{1}{y\gamma^L}.$$

(25) can be rewritten as

$$P(\mathcal{T}_1) \doteq \frac{1}{\gamma^L}\int_{y=\frac{1}{\gamma}}^{\infty} \frac{e^{-y}}{y}\, dy. \tag{26}$$

Since $\frac{1}{\gamma} \to 0$, the exponential integral is given by [12],

$$\int_{y=\frac{1}{\gamma}}^{\infty} \frac{e^{-y}}{y}\, dy = 0.5772 + \ln(\gamma) + o\left(\frac{1}{\gamma}\right).$$

Then, we have for the area $\mathcal{T}_1$

$$P(\mathcal{T}_1) \doteq \frac{\ln(\gamma)}{\gamma^L} \doteq \frac{1}{\gamma^L}. \tag{27}$$

Calculation of $P(\mathcal{T}_2)$:

$$P(\mathcal{T}_2) = \int_{y=0}^{\frac{1}{\gamma}} e^{-y}\, dy \int_{x=\frac{1}{\gamma^{(L-1)}}}^{\frac{1}{y\gamma^L}} p_X(x)\, dx.$$

$\forall\epsilon > 0$, let $L_0 = L - \epsilon$ and $y_0 = \frac{1}{\gamma^{L_0}}$ then $P(\mathcal{T}_2)$ is given by

$$\begin{aligned} P(\mathcal{T}_2) &= \int_{y=0}^{y_0} e^{-y}\, dy \int_{x=\frac{1}{\gamma^{(L-1)}}}^{\frac{1}{y\gamma^L}} p_X(x)\, dx \\ &+ \int_{y_0}^{\frac{1}{\gamma}} e^{-y}\, dy \int_{x=\frac{1}{\gamma^{(L-1)}}}^{\frac{1}{y\gamma^L}} p_X(x)\, dx \end{aligned} \tag{28}$$

The upper limit of the inner integral goes to 0 because

$$\frac{1}{y\gamma^L} \le \frac{1}{y_0\gamma^L} = \frac{1}{\gamma^{(L-L0)}} = \frac{1}{\gamma^\epsilon}.$$

In the first term of (28), the inner integral is upperbounded by 1. In the second term of (28), the inner integral is found by applying (22) twice. We reach an upper bound for $P(\mathcal{T}_2)$,

$$P(\mathcal{T}_2) \le T_2 \tag{29}$$

where

$$\begin{aligned} T_2 &\doteq \int_{y=0}^{y_0} e^{-y}dy + \int_{y_0}^{\frac{1}{\gamma}} e^{-y}\left[\frac{1}{y\gamma^L} - \frac{1}{\gamma^{L-1}}\right] dx \\ &\le y_0 + \frac{1}{\gamma^L}\int_{y_0}^{\frac{1}{\gamma}} \frac{e^{-y}}{y}\, dy \ \dot{\le}\ \frac{1}{\gamma^{L-\epsilon}} + \frac{1}{\gamma^L}\int_{y_0}^{\frac{1}{\gamma}} \frac{e^{-y}}{y}\, dy\ . \end{aligned} \tag{30}$$

The evaluation of the exponential integral in (30) gives [12]

$$\int_{y_0}^{\frac{1}{\gamma}} \frac{e^{-y}}{y}\, dy = (L_0 - 1)\ln(\gamma) + o\left(\frac{1}{\gamma}\right)\ . \tag{31}$$

Then, from (29), (30), and (31), $\forall\epsilon > 0$

$$P(\mathcal{T}_2) \dot{\le} \frac{1}{\gamma^{L-\epsilon}} + \frac{(L-\epsilon-1)\ln(\gamma)}{\gamma^L} \doteq \frac{1}{\gamma^{L-\epsilon}} \tag{32}$$

Using (24), (27), and (32), we get the upper bound

$$P_{out}(\gamma) \dot{\le} \frac{1}{\gamma^L}\ . \tag{33}$$

Similar to the case of $L = 2$, we conclude from the lower bound and the upper bound derived above that

$$P_{out}(\gamma) \ \doteq\ \frac{1}{\gamma^L}\ .$$

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V. Simulation Results

To illustrate Poltyrev outage limit and its great advantage for lattice decoding, we show computer simulations for low-density lattices in dimension 64. We use the low-density lattices by Boutros in [13]. Particularly, we use a full-diversity LDLC (FD-LDLC) proposed in Theorem 1 of [13] that achieves full diversity under ML decoding for $L = 2$.

The integer-check matrix H of such LDLC is described as follows. Let $H = [h_{ij}]$ be the $n \times n$ parity-check matrix of a real lattice Λ of even rank n, where $h_{ij} \in \mathbb{Q}$, the field of rationals. Let us decompose H into four $n/2 \times n/2$ submatrices as follows.

$$H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Assume that A, B, C, and D have full rank. Let θ_1 and θ_2 be two algebraic numbers of degree ≥ 2 such that $\theta_2/\theta_1 \notin \mathbb{Q}$. Then, the two lattices defined respectively by the integer-check matrices

$$\begin{bmatrix} \theta_1 A & \theta_1 B \\ \theta_2 C & \theta_2 D \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \theta_1 A & \theta_2 B \\ \theta_2 C & \theta_1 D \end{bmatrix} \tag{34}$$

are full-diversity lattices under ML decoding. We utilize the integer-check matrix with dimension $n = 64$ constructed according to the second matrix of (34) for our simulation where we select $\theta_1 = 1$ and $\theta_2 = \sqrt{2}$.

We do not use any shaping region for the selected LDLC. The infinite LDLC constellation is decoded using the sphere decoding algorithm [14]. The outage boundary defining $P_{out}(\gamma)$ is used to detect inadmissible fading coefficients. If the channel state is under the outage boundary, an error is declared without running the sphere decoder. This leads to a drastic reduction in the decoding runtime (figures are not shown in this paper due to lack of space). As mentioned before, it would have been impossible to decode the FD-LDLC in dimension 64 without establishing $P_{out}(\gamma)$. Computer simulation results are illustrated in Fig.2. Double diversity is clearly observed for $P_{out}(\gamma)$ and the FD-LDLC infinite constellation.

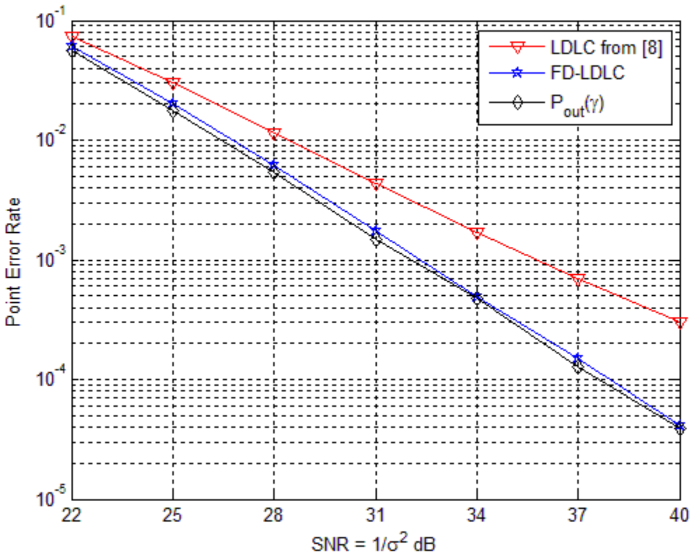

Fig. 2: ML decoding of double-diversity LDLC in dimension $n = 64$. The plot also shows Poltyrev outage limit and a standard LDLC.

VI. Conclusion

In this paper, we defined a Poltyrev outage limit for lattices in presence of block fading. We proved that its diversity order is equal to the number of degrees of freedom in the BF channel. An important application is lattice decoding of low-density lattice codes. The outage boundary of the newly defined Poltyrev outage limit is used to declare an outage error without decoding, which makes tractable the decoding of low-density lattices on the BF channel.

Acknowledgment

This publication was made possible by NPRP grant number 5-600-2-242 from the Qatar National Research Fund, a member of Qatar Foundation. The work of Ezio Biglieri was supported by Project TEC2012-34642.

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