Spatial Coupling of Root-LDPC: Parity Bits Doping

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***Abstract*—Random low-density parity-check (LDPC) ensembles do not achieve full diversity on block-fading channels. To cope with quasi-static fading, a special LDPC structure, known as root-LDPC, has been introduced. A root-LDPC ensemble guarantees that any information bit receives messages from all channel states. Results in the literature show that the gap between the root-LDPC boundary and the capacity boundary in the fading plane is not small enough. In this paper, we propose to saturate the whole root-LDPC boundary via spatial coupling. For simplicity, we adopt an equivalent erasure channel model rather than a block fading model. It is shown that spatial coupling of parity bits only is sufficient to saturate the root-LDPC threshold boundary in the erasure plane.**

I. Introduction

The theory of error-correcting codes was limited to channels with errors and erasures that are independently located inside a codeword [1], i.e. ergodic channels. Similarly, all channels considered in modern coding theory are also ergodic in nature [2][3][4]. The non-ergodic fading channel encountered in wireless communications requires a special coding approach [5][6]. Root-LDPC codes are low-density parity-check codes specifically designed to tackle with non-ergodic (quasi-static) fading [7][8]. The key idea is to bring more structure to a random LDPC ensemble in order to let information bit nodes receive belief-propagation (BP) messages including all available fading states. Hence, a root-LDPC ensemble attains a diversity order which is equal to the number of degrees of freedom in the block-fading channel.

Recently, spatial coupling was shown to saturate the BP threshold of LDPC ensembles on binary-input memoryless channels [9][10]. The technique of forward layered coupling, described in [11], shows a simple coupling method where local edges and coupling edges are treated separately. In this paper, we propose to apply forward layered coupling to root-LDPC codes and study the effect of spatial coupling on their outage boundary in the fading plane. For the sake of simplicity, the fading plane approach is replaced by its equivalent counterpart, the erasure plane approach. The interest in applying spatial coupling to root-LDPC codes is motivated by the use of anti-root LDPC ensembles for secure communications on multiple-link channels [12][13].

The paper is organised as follows. Section II presents a very brief summary of the root-LDPC code structure. The reader is assumed to have a minimal background on coding for fading channels, mainly the notion of diversity [5][6]. The density evolution (DE) analysis of uncoupled root-LDPC is presented in Section III. To improve the threshold boundary of root-LDPC, we propose a new spatial coupling structure based on parity bits doping in Section IV. Section V presents the threshold saturation result achieved by coupled root-LDPC ensembles. The conclusions are finally drawn in Section VI.

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II. Root-LDPC codes

Root-LDPC ensembles are multi-edge-type LDPC ensembles with specific properties [7]. For simplicity, let us consider a rate-$1/2$ LDPC code of length N, where N bits are transmitted on a block-fading channel with $n_c = 2$ fading coefficients per codeword. We define four classes of bits and two classes of checks. $N/4$ information bits $1i$ and $N/4$ parity bits $1p$ are transmitted on fading coefficient α_1, whereas $N/4$ information bits $2i$ and $N/4$ parity bits $2p$ are transmitted on fading coefficient α_2. The design of root-LDPC codes is based on a special type of checknode called *rootcheck*. A rootcheck of type 1 for a bit transmitted on fading α_1 is a check where all other connected bits are transmitted on α_2. The rootchecks of type 2 are defined similarly. In order to guarantee full diversity for information bits, information and parity bits are connected to checks $1c$ and $2c$ as shown by the compact Tanner graph of the rate-$1/2$ root-LDPC ensemble in Figure 1.

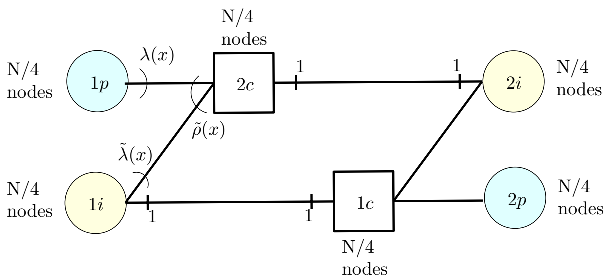

Figure 1. Compact Tanner graph representation for a rate-$1/2$ root-LDPC ensemble. If the bits transmitted on one fading are erased, the erased information bits are recovered.

The rate-$1/2$ root-LDPC code described by the compact Tanner graph in Figure 1 is full-diversity under iterative message passing when transmitted on a 2-state block-erasure channel. This root-LDPC code is also full-diversity under iterative BP decoding when transmitted on a 2-state block-fading channel [7]. This code is also MDS according to the block-fading Singleton bound and has the highest coding rate that attains a double diversity (see Sec. V in [8]).

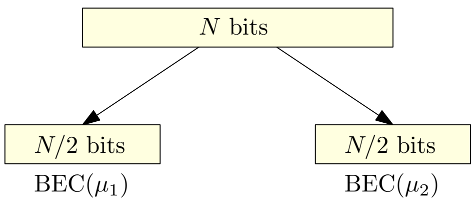

Figure 2. Parallel binary erasure channels where half of the bits are transmitted through BEC(μ_1) and the the other half of the bits are transmitted through BEC(μ_2).

III. Density evolution of root-LDPC ensembles

This section summarises the DE equations of the information and parity messages on two parallel BECs for the uncoupled root-LDPC ensembles [7]. Consider the channel model in Figure 2, where N bits are transmitted on two parallel BECs with the erasure probabilities μ_1 and μ_2. $N/2$ bits are transmitted on BEC(μ_1) and $N/2$ bits are transmitted on BEC(μ_2).

Define the degree distribution polynomials as

$$\lambda(x) = \sum_{i=2}^{d_b} \lambda_i x^{i-1} \qquad \rho(x) = \sum_{j=2}^{d_c} \rho_j x^{j-1} \tag{1}$$

The definitions of the degree polynomials $\mathring{\lambda}(x)$, $\mathring{\rho}(x)$, $\tilde{\lambda}(x)$, and $\tilde{\rho}(x)$ are as in [7]. Let us define the six message densities for a root-LDPC ensemble as follows:

- f_1 is the density of message $1i \to 2c$.
- q_1 is the density of message $1i \to 1c$.
- f_2 is the density of message $2i \to 1c$.
- q_2 is the density of message $2i \to 2c$.
- g_1 is the density of message $1p \to 2c$.
- g_2 is the density of message $2p \to 1c$.

DE equations for the six message densities q_1^m, f_1^m, g_1^m, q_2^m, f_2^m, and g_2^m can be found by drawing the local neighborhood of each type of bitnodes at decoding iteration $m+1$ as:

$$q_1^{m+1} = \mu_1 \mathring{\lambda}\left(1-(1-q_2^m)\,\tilde{\rho}\left(1-f_e f_1^m - g_e g_1^m\right)\right) \tag{2}$$

$$f_1^{m+1} = \mu_1 \tilde{\lambda}\left(1-(1-q_2^m)\,\tilde{\rho}\left(1-f_e f_1^m - g_e g_1^m\right)\right)\left(1-\mathring{\rho}\left(1-f_e f_2^m - g_e g_2^m\right)\right) \tag{3}$$

$$g_1^{m+1} = \mu_1 \lambda\left(1-(1-q_2^m)\,\tilde{\rho}\left(1-f_e f_1^m - g_e g_1^m\right)\right) \tag{4}$$

$$q_2^{m+1} = \mu_2 \mathring{\lambda}\left(1-(1-q_1^m)\,\tilde{\rho}\left(1-f_e f_2^m - g_e g_2^m\right)\right) \tag{5}$$

$$f_2^{m+1} = \mu_2 \tilde{\lambda}\left(1-(1-q_1^m)\,\tilde{\rho}\left(1-f_e f_2^m - g_e g_2^m\right)\right)\left(1-\mathring{\rho}\left(1-f_e f_1^m - g_e g_1^m\right)\right) \tag{6}$$

$$g_2^{m+1} = \mu_2 \lambda\left(1-(1-q_1^m)\,\tilde{\rho}\left(1-f_e f_2^m - g_e g_2^m\right)\right) \tag{7}$$

where the multi-edge-type fractions are defined as functions of the average bitnode degree $\bar{d}_b$:

$$f_e = 1 - g_e = \frac{\bar{d}_b - 1}{2\bar{d}_b - 1} \tag{8}$$

For regular ensembles with bitnode degree d_b and check node degree d_c, $\mathring{\lambda}(x) = \lambda(x)$, $\mathring{\rho}(x) = \rho(x)$, $\tilde{\lambda}(x) = \lambda(x)/x$, and $\tilde{\rho}(x) = \rho(x)/x$. Consequently, $q_1(x) = g_1(x)$ and $q_2(x) = g_2(x)$. Then, we have four density evolution equations. And $\lambda(x)$ and $\rho(x)$ are monomials:

$$\lambda(x) = x^{d_b-1} \qquad \rho(x) = x^{d_c-1} \tag{9}$$

Then, DE equations for BEC(μ_1) and BEC(μ_2) can be written as:

$$q_1^{m+1} = \mu_1 \left(1-(1-q_2^m)\left(1-f_e f_1^m - g_e q_1^m\right)^{d_c-2}\right)^{d_b-1} \tag{10}$$

$$f_1^{m+1} = \mu_1 \left(1-(1-q_2^m)\left(1-f_e f_1^m - g_e q_1^m\right)^{d_c-2}\right)^{d_b-2}\left(1-\left(1-f_e f_2^m - g_e q_2^m\right)^{d_c-1}\right) \tag{11}$$

$$q_2^{m+1} = \mu_2 \left(1-(1-q_1^m)\left(1-f_e f_2^m - g_e q_2^m\right)^{d_c-2}\right)^{d_b-1} \tag{12}$$

$$f_2^{m+1} = \mu_2 \left(1-(1-q_1^m)\left(1-f_e f_2^m - g_e q_2^m\right)^{d_c-2}\right)^{d_b-2}\left(1-\left(1-f_e f_1^m - g_e q_1^m\right)^{d_c-1}\right) \tag{13}$$

Recall that for root-LDPC, only the information bits are connected to rootchecks and have full-diversity. Thus, the threshold bound is determined by the convergence of the densities f_1 and f_2 associated with the messages $1i \to 2c$ and $2i \to 1c$ respectively.

As we show in Section V, the gap between the threshold boundary of root-LDPC and the capacity boundary of the channel is not small enough. In [11], a forward layered spatial coupling technique is proposed to improve the threshold boundary of random LDPC ensembles. In the next section, we apply the forward layered spatial coupling to root-LDPC ensembles in order to improve the coding gain and saturate the threshold bound.

IV. Spatial coupling of root-LDPC ensembles

In this section, in order to improve the coding gain, we propose a spatial coupling structure for root-LDPC based on the coupling structure in [11] for random LDPC ensembles.

Construction of the partially-coupled root-LDPC ensemble: Our spatial coupling structure is similar to [11]. L copies of (d_b, d_c) root-LDPC ensembles are placed in spatial positions $l = 1, 2, ..., L$ to form a chain of length L. Then, extra edges are added to couple the ensembles. In order to simplify the construction, only parity bitnodes are coupled to the checknodes in the next spatial position resulting in a partial-coupling scheme as shown in Figure 3. To our knowledge, this is the first time partial coupling is applied to LDPC ensembles for parity bits doping. For termination at the right end, extra check nodes are placed at the $(L+1)$-Th. spatial position.

In addition to the local message densities defined in Section III, we have four more coupling messages. Let us define the coupling message densities as follows:

- n_1 is the density of message $1p(l) \to 2c(l+1)$.
- n_2 is the density of message $2p(l) \to 1c(l+1)$.
- k_1^{l+1} is the density of message propagating backward $2c(l+1) \to 1p(l)$.

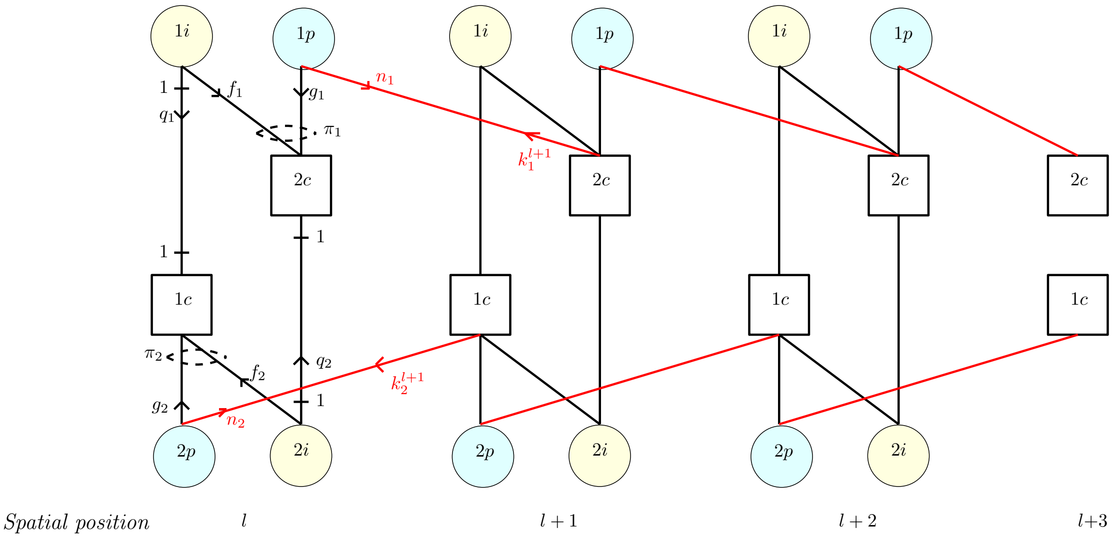

Figure 3. Forward layered spatial coupling of root-LDPC. The chain is terminated at the right end by the checks placed at the spatial position $L+1$. Since only the parity bits are coupled to the checks, this is a partial coupling scheme.

- k_2^{l+1} is the density of message propagating backward $1c(l+1) \to 2p(l)$.

Proposition 1: Let (d_b, d_c) denotes the degrees of bitnodes and checknodes of the uncoupled regular LDPC ensemble and (d_{bs}, d_{cs}) denotes the degrees of the extra edges for coupling the parity bitnodes and the checks. Then, DE equations for forward layered spatially coupled root-LDPC ensemble are given by (14) – (18) for bit node types $1i$ and $1p$. DE equations for bit node types $2i$ and $2p$ can be derived by changing the node types and indices accordingly.

- ***Local messages***

$$q_1^l = \mu_1 \Big(1 - \left(1 - q_2^l\right)\left(1 - f_e f_1^l - g_e g_1^l\right)^{d_c-2} \left(1 - n_1^{l-1}\right)^{d_{cs}}\Big)^{d_b-1} \quad (14)$$

$$f_1^l = \mu_1 \Big(1 - \left(1 - q_2^l\right)\left(1 - f_e f_1^l - g_e g_1^l\right)^{d_c-2} \left(1 - n_1^{l-1}\right)^{d_{cs}}\Big)^{d_b-2} \Big(1 - \left(1 - f_e f_2^l - g_e g_2^l\right)^{d_c-1}\left(1 - n_2^{l-1}\right)^{d_{cs}}\Big) \quad (15)$$

$$g_1^l = \mu_1 (k_1^{l+1})^{d_{bs}} \Big(1 - \left(1 - q_2^l\right)\left(1 - f_e f_1^l - g_e g_1^l\right)^{d_c-2} \left(1 - n_1^{l-1}\right)^{d_{cs}}\Big)^{d_b-1} \quad (16)$$

- ***Coupling messages (forward)***

$$n_1^l = \mu_1 (k_1^{l+1})^{d_{bs}-1} \Big(1 - \left(1 - q_2^l\right)\left(1 - f_e f_1^l - g_e g_1^l\right)^{d_c-2} \left(1 - n_1^{l-1}\right)^{d_{cs}}\Big)^{d_b} \quad (17)$$

- ***Coupling messages (backward)***

$$k_1^{l+1} = 1 - \left(1 - q_2^{l+1}\right)\left(1 - f_e f_1^{l+1} - g_e g_1^{l+1}\right)^{d_c-1} \left(1 - n_1^{l}\right)^{d_{cs}-1} \quad (18)$$

Proof: A reader who is familiar with the modern coding theory [4] can derive the DE equations from the local tree neighbourhood of the nodes shown in Figure 4 – Figure 8. ■

Since information bits are connected to rootchecks, the diversity order of the information bits is 2. To improve the coding gain of the information bits, we create more parity bits of diversity order 2 by using a root-LDPC(2π) family instead of a root-LDPC(4π) family (for more information about the root-LDPC(2π) and root-LDPC(4π) families refer to [8]).

Similar to root-LDPC, the threshold boundary of coupled root-LDPC is determined by the convergence of the densities associated with the local messages $1i(l) \to 2c(l)$ and $2i(l) \to 1c(l)$. In fact, we have L layered threshold boundaries corresponding to L copies of the root-LDPC in the chain.

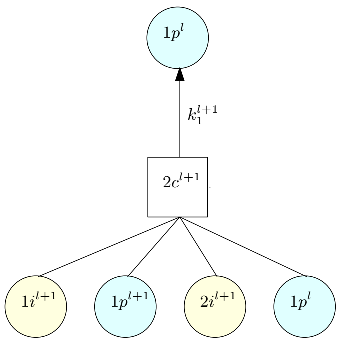

Figure 4. Local tree neighbourhood of checknode $2c$. This tree is used to determine the evolution of the backward coupling message k_1^{l+1}.

V. RESULTS

In this section, we present the outage boundaries of rate-$1/2$ random LDPC ensembles, uncoupled root-LDPC ensembles, and coupled root-LDPC ensembles in the erasure plane for two parallel erasure channels BEC(μ_1) and BEC(μ_2) shown in Figure 2.

The capacities of the erasure channels BEC(μ_1) and BEC(μ_2) are $C_1 = 1 - \mu_1$ and $C_2 = 1 - \mu_2$ respectively. When

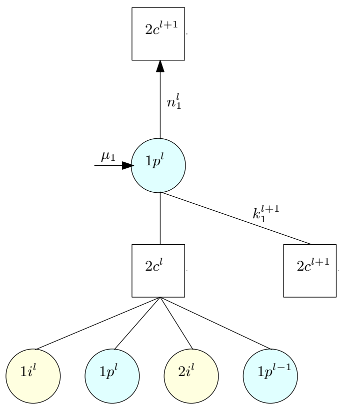

Figure 5. Local tree neighbourhood of bitnode $1p$. This tree is used to determine the evolution of the forward coupling message n_1^l.

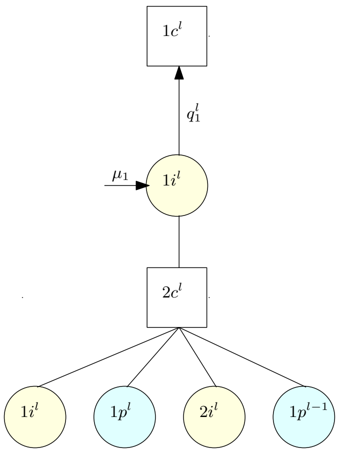

Figure 6. Local tree neighbourhood of bitnode $1i$. This tree is used to determine the evolution of the local message q_1^l.

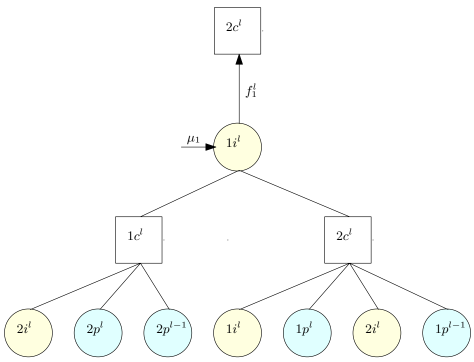

Figure 7. Local tree neighbourhood of bitnode $1i$. This tree is used to determine the evolution of the local message f_1^l.

the two parallel channels are used uniformly, the capacity is the average of C_1 and C_2

$$C = \frac{1}{2}(C_1 + C_2) = 1 - \frac{1}{2}(\mu_1 + \mu_2) \tag{19}$$

For rate-$1/2$, the capacity limit is achieved when $C = R = 1/2$. Then, the capacity bound of the channel is given by the straight line $\mu_1 + \mu_2 = 1$.

Consider a random $(3,6)$-LDPC ensemble. When $\mu_1 = \mu_2$, we have the standard DE equation for the $(3,6)$-LDPC

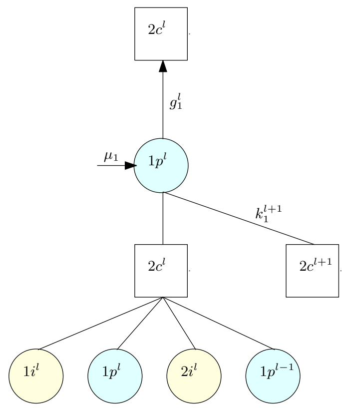

Figure 8. Local tree neighbourhood of bitnode $1p$. This tree is used to determine the evolution of the local message g_1^l.

ensemble

$$x = \mu \left(1 - (1-x)^5\right)^2 \tag{20}$$

and the BP threshold for $\mu_1 = \mu_2 = \mu$ is 0.429. Since the LDPC ensemble is random, a bit is transmitted through BEC(μ_1) or BEC(μ_2) equiprobably. So, the random LDPC is undergoing an average channel BEC($(\mu_1+\mu_2)/2$). Then, the corresponding DE equation is

$$x = \frac{\mu_1 + \mu_2}{2} \left(1 - (1-x)^5\right)^2 \tag{21}$$

and the ergodic threshold for the random $(3,6)$-LDPC ensemble is given by the line $(\mu_1 + \mu_2)/2 = 0.429$. Similarly, the ergodic threshold for the random $(4,8)$-LDPC ensemble is given by the line $(\mu_1 + \mu_2)/2 = 0.383$.

In the erasure plane, the (μ_1, μ_2) points under the threshold boundary cause the DE equations to converge and the (μ_1, μ_2) points above the threshold boundary result in the divergence of the DE equations. Thus, outage occurs when the (μ_1, μ_2) points are above the threshold boundary.

The threshold boundary of the uncoupled $(3,6)$ root-LDPC is shown Figure 9. The threshold boundary of the uncoupled $(3,6)$ root-LDPC is close to the ergodic threshold of the random $(3,6)$-LDPC except for the cases where $\mu_1 = 0$ or $\mu_2 = 0$. As the uncoupled root-LDPC has full diversity, $(\mu_1, \mu_2) = (1,0)$ and $(\mu_1, \mu_2) = (0,1)$ are on the threshold boundary of the uncoupled $(3,6)$ root-LDPC. However, the gap between the threshold boundary of the uncoupled $(3,6)$ root-LDPC and the capacity boundary of the channel is not small enough.

The threshold boundary improvement achieved by the proposed spatial coupling scheme is presented in Figure 10 for chain length $L = 60$. The chain length should be large enough, i.e. $L \geq 50$, so that the spatial coupling effect can be observed and the rate converges to the targetted rate. For larger values of L, the threshold boundary does not change. For simplicity, the parity bitnodes and checknodes are coupled with single edges, i.e. $d_{bs} = d_{cs} = 1$. Although we have L threshold boundaries for L copies of the root-LDPC in the chain, the threshold boundaries are very close to each other. Thus, we plot a single boundary.

Figure 10 shows the threshold boundary of coupled (4,8) root-LDPC. Although only a partial coupling scheme is applied

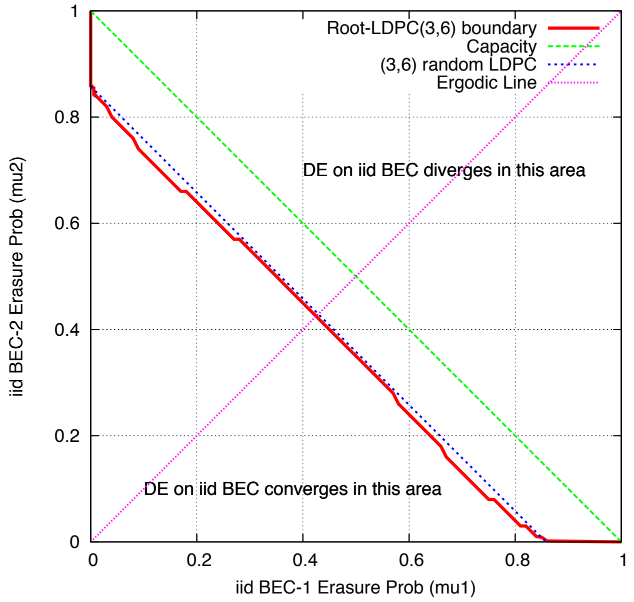

Figure 9. BEC threshold boundaries for random $(3, 6)$-LDPC and uncoupled $(3, 6)$ root-LDPC ensembles in the erasure plane.

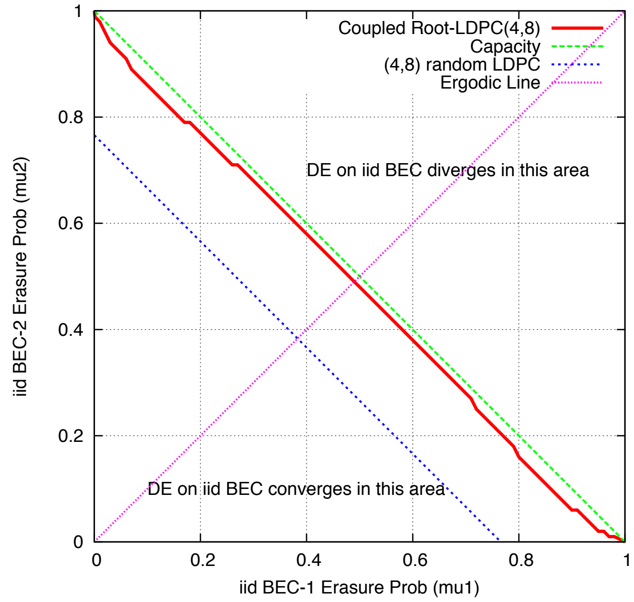

Figure 10. Saturation of the threshold boundary after coupling parity bits. Threshold boundaries for random $(4, 8)$-LDPC and coupled $(4, 8)$ root-LDPC ensembles are shown in the erasure plane.

by parity bits doping, the threshold boundary is very close to the capacity boundary of the channel. This significant threshold saturation is due to the coding gain introduced by spatial coupling. Figure 10 also shows that the coupled $(4, 8)$ root-LDPC ensemble has full diversity.

Note that, random rate-$1/2$ LDPC ensembles cannot achieve full diversity [8], which means that when all the bits transmitted through BEC(μ_1) are erased, it is not possible to recover these erased bits from the bits transmitted through BEC(μ_2) and vice versa. Thus, the ergodic threshold boundary lines for random LDPC ensembles do not pass through the points $(1, 0)$ and $(0, 1)$ in the erasure plane as shown in Figure 9 and Figure 10.

Increasing the degrees of the bitnodes and checknodes deteriorates the threshold boundary of the random LDPC under BP decoding as presented in Figure 9 and Figure 10.

VI. Conclusions

A novel spatial coupling scheme is proposed for root-LDPC ensembles. Local protographs are coupled by new edges connected to their parity bits only. As expected from parity bits doping [8], coupling parity bits is greatly enhancing the coding gain of information bits. The spatially-coupled root-LDPC ensembles achieve threshold boundaries very close to the capacity boundary (saturation) in the erasure plane. This behaviour in the erasure plane will automatically result in a saturation of the root-LDPC outage boundary on a real block-fading channel with additive noise.

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