Diversity of Low-Density Lattices

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***Abstract*—We describe full-diversity constructions of real lattices defined by their integer-check matrix, i.e. the inverse of their generator matrix. In the first construction suited to maximum-likelihood decoding, these lattices are defined by sparse (low-density) or non-sparse integer-check matrices. Based on a special structure of the lattice binary image, a second full-diversity lattice construction is described for sparse integer-check matrices in the context of iterative probabilistic decoding. Full diversity is theoretically proved in both cases. Computer simulation results also confirm that the proposed low-density lattices attain the maximal diversity order.**

I. Introduction

Error-correcting codes and modulation schemes are integral parts of communication systems. Coded modulations which combine these two functionalities were extensively studied in the literature [1, Sec. 8.12]. Some coded modulation schemes rely on infinite constellations (ICs). ICs are structures in the n-dimensional Euclidean space that have no power constraints and hence cannot be used directly for transmission over power-constrained channels. Practical coded modulation schemes derived from ICs use only those points of the constellation which lie inside some specific region (known as *shaping region*) in the Euclidean space.

An infinite lattice constellation, also known as a *lattice* [2], is a constellation with a strong algebraic structure. A lattice is a discrete additive sub-group of the real space. A lattice codebook can be constructed from a lattice using a shaping region which could be the Voronoi region of another lattice or a sub-lattice [3]. *Low-density lattice codes* (LDLCs) are a special class of lattices proposed by Sommer *et al.* in [4] which can be decoded by iterative probabilistic message passing. Another family of iteratively decodable lattices has been recently published in [5], where authors described a generalized low-density construction. In this paper, we focus on LDLC without using any shaping region.

We consider transmission using lattices over a general non-ergodic fading (block-fading) channels for single-input single-output systems. For such a channel model a codeword of length n is divided into L blocks of equal length n/L such that the fading within a given block is same whereas it is different and independent across different blocks. Such a channel has L degrees of freedom. If a lattice code is used for transmission over such a block-fading channel and the error probability at the output of the decoder is proportional to $1/\gamma^L$ (where γ is the signal-to-noise ratio) then such a lattice code is said to have diversity order L or referred to as full-diversity lattice. However, randomly constructed lattice codes does not have full-diversity property. To the best of the authors knowledge no effort has been in the literature to design full-diversity lattice codes. Infinite constellations over multiple-antenna fading channels are studied in [6].

In this paper we propose methods to construct LDLC which exhibit full-diversity when decoded by maximum-likelihood (ML) and iterative decoder. In first part, we give conditions for a lattice to be full-diversity under ML decoding and then propose full-diversity construction method for lattice codes. This construction method can be used to generate sparse and non-sparse lattice codes suitable for ML decoding. It has been shown theoretically that such lattice codes indeed have full-diversity property. In this second part, first the random LDLC is analyzed through the low-density parity-check code derived from binary image of LDLC. Using results from this analysis, a method to construct full-diversity LDLC has been proposed. Further, we also prove that such LDLC does exhibit full-diversity property when decoded by the iterative probabilistic decoding algorithm. Some simulation results for double-diversity LDLC constructed for ML decoding and iterative decoding are also provided which validate our theoretical results.

The rest of the paper is structured as follows; the communication model considered in this paper is described in section II. Section III propose the full-diversity lattice code suitable for ML decoding. Full-diversity LDLC suitable for iterative decoding are proposed in section IV. Section V discuss simulation results. We conclude in Section VI.

II. Communication Model

We use $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{Q}$ to denote the field of real numbers, ring of integers and the field of fractions of $\mathbb{Z}$ respectively. A a point lattice or simply *lattice* $\Lambda \subset \mathbb{R}^n$ is a free $\mathbb{Z}$-module of rank n in $\mathbb{R}^n$. A *point* is an element that belongs to Λ. Any point $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^T \in \Lambda$ can be written as an integer linear combination of n points $\boldsymbol{x} = \sum_{i=1}^{n} z_i \boldsymbol{v}_i$, where $\{\boldsymbol{v}_i\}$ is a $\mathbb{Z}$-basis of Λ, $v_{i,j} \in \mathbb{R}$, and $z_i \in \mathbb{Z}$. A *generator matrix* G for Λ is $n \times n$ matrix built from a basis $\{\boldsymbol{v}_i\}, i \in \{1, \dots, n\}$. In column convention, let $G = [v_{i,j}]$, then a lattice point is written as $\boldsymbol{x} = G\boldsymbol{z}$, where $\boldsymbol{z} \in \mathbb{Z}^n$. The *volume of the lattice* Λ is given by $|\det(G)|$. For more information on lattices, we refer the reader to [2].

A *lattice code* of dimension n can be defined using a lattice Λ and a shaping region $S \subset \mathbb{R}^n$ where the codewords for the lattice code are the lattice points that lie within the shaping region S. The integer-check matrix for such a code is given by $H = G^{-1} = [h_{i,j}]$. For lattice codes, the integer-check equation for a codeword or lattice point $\boldsymbol{x}$ is given by

$$\sum_j h_{i,j} \; x_i \in \mathbb{Z}, \forall i. \tag{1}$$

The notion of code rate and therefore channel capacity is not useful for lattices as the signal power can be made arbitrarily large. For lattices used for transmission over additive white Gaussian noise channel without restrictions, Poltyrev [7] showed that there exists a lattice that can be correctly decoded if and only if the noise variance is less than a threshold. We refer to this threshold as *Poltyrev threshold*. Formally, Poltyrev threshold can be defined as follows: There exists a lattice Λ of high enough dimension-n for which the transmission error probability over AWGN channel can be reduced to an arbitrary low level if and only if $\sigma^2 < \sigma^2_{\max}$ [7][4] where σ^2 is the noise variance per dimension and Poltyrev threshold $\sigma^2_{\max}$ is given by

$$\sigma^2_{max} = \frac{|\det(G)|^{\frac{2}{n}}}{2\pi e}. \tag{2}$$

The integer-check matrix of an LDLC considered in this paper is a sparse matrix where $|\det(H)| = |\det(G)| = 1$. The corresponding iterative decoding algorithm whose complexity is linear in block length is also proposed in [4]. More information about LDLC can be found in [4].

The communication system considered in this paper is assumed to have coherent detection along with perfect channel state information at the receiver side only. We assume that the fading channel is flat and hence without multiple paths [1]. Fading coefficients are assumed to be Rayleigh distributed and are real non-negative, $0 \leq \alpha < +\infty$, $p(\alpha) = 2\alpha \exp(-\alpha^2)$. We consider non-ergodic fading where the fading coefficients will take L values only within a lattice point, usually $L \ll n$. Let Λ be a lattice of rank n in $\mathbb{R}^n$ and let x be a lattice point then the general non-ergodic (block-fading) channel model with diversity L is

$$y_i = \alpha_j x_i + \eta_i, \quad j = \left\lceil \frac{i}{n/L} \right\rceil, \quad i = 1, 2, \ldots n, \tag{3}$$

The fadings $\alpha_1, \alpha_2, \ldots, \alpha_L$ are independent and identically distributed (i.i.d.) Rayleigh distributed fading coefficients and $\eta_i \sim \mathcal{N}(0, \sigma^2)$.

Let γ be the signal-to-noise ratio (SNR) for an infinite lattice constellation,

$$\gamma = \frac{|\det(G)|^{\frac{2}{n}}}{\sigma^2}. \tag{4}$$

Definition 1. *Consider a fading channel with L independent fading coefficients per lattice point. Λ is a full-diversity lattice under ML decoding if the point error probability $P_e(\Lambda)$ at the ML decoder output is proportional to $\frac{1}{\gamma^L}$, for $\gamma = \frac{1}{\sigma^2} >> 1$.*

Recently, the authors in [8] introduced a *Poltyrev outage limit* for infinite lattice constellations used over block-fading (BF) channels. Poltyrev outage limit is defined as follows [8]: if a given lattice Λ is used for transmission over a BF channel with coefficients $\alpha_1, \ldots, \alpha_L$ then the lattice decoder can decode correctly if and only if $\sigma^2 < \sigma^2_{\max}(\alpha_1, \ldots, \alpha_L)$. Otherwise, if $\sigma^2 > \sigma^2_{\max}(\alpha_1, \ldots, \alpha_L)$ an outage occurs. For a fixed fading, the noise variance defining Poltyrev limit is

$$\sigma^2_{max}(\alpha_1, \ldots, \alpha_L) = \frac{\prod_{l=1}^{L} \alpha_l^{2/L} |\det(G)|^{\frac{2}{n}}}{2\pi e}. \tag{5}$$

III. Full-Diversity Construction of LDLC under ML decoding

Let Λ be a real lattice of rank n defined by a $n \times n$ integer-check matrix H. Assume that n is multiple of L, where L is the diversity order of the block-fading channel. Let us write H in the form

$$H = \left[\; \tilde{H}_1 \mid \tilde{H}_2 \mid \; \ldots \; \mid \tilde{H}_L \; \right], \tag{6}$$

where $\tilde{H}_j$ is a $n \times n/L$ matrix, $j = 1 \ldots L$. In the above expression of the integer-check matrix H, the channel is assumed to have the same fading value α_j affecting all n/L lattice components associated to the columns of $\tilde{H}_j$, as defined in (3) in Sec. II. Using the L submatrices $\tilde{H}_j$, let us build a new shortened integer-check matrix Ψ_k of size $n \times \ell n/L$ by combining ℓ submatrices out of L, for $\ell = 1 \ldots L-1$. The number of shortened integer-check matrices is

$$K = \sum_{\ell=1}^{L-1} \binom{L}{\ell} = 2^L - 2. \tag{7}$$

Also, for $k = 1 \ldots K$, we define the function $\kappa(k)$ as

$$\kappa(k) = i \times \frac{n}{L} \tag{8}$$

for i that satisfies

$$\sum_{\ell=1}^{i-1} \binom{L}{\ell} \leq k \leq \sum_{\ell=1}^{i} \binom{L}{\ell}, \tag{9}$$

such that Ψ_k is a $n \times \kappa(k)$ matrix.

For example, for $L = 2$, we have $\Psi_1 = \tilde{H}_1$, $\Psi_2 = \tilde{H}_2$, and $\kappa(1) = \kappa(2) = n/L$. For $L = 3$, we have $\Psi_1 = \tilde{H}_1$, $\Psi_2 = \tilde{H}_2$, $\Psi_3 = \tilde{H}_3$, $\Psi_4 = [\tilde{H}_1 | \tilde{H}_2]$, $\Psi_5 = [\tilde{H}_1 | \tilde{H}_3]$, $\Psi_6 = [\tilde{H}_2 | \tilde{H}_3]$, $\kappa(1) = \kappa(2) = \kappa(3) = n/L$, and $\kappa(4) = \kappa(5) = \kappa(6) = 2n/L$.

Definition 2. *Consider a fading channel with L independent fading coefficients per lattice point. Λ is a full-diversity lattice under ML decoding if the point error probability $P_e(\Lambda)$ at the ML decoder output is proportional to $\frac{1}{\gamma^L}$, for $\gamma = \frac{1}{\sigma^2} >> 1$.*

The purpose of this paper is to build low-density lattices such that $P_e \approx K_e/\gamma^L$ at large signal-to-noise ratio, where K_e is a non-negative real constant. The so-called coding gain [9] is given by $1/\sqrt[L]{K_e}$. Maximizing the coding gain or equivalently minimizing K_e is not the subject of this paper. Following the above definition and using the K shortened integer-check matrices, we can now state a simple lemma useful for proving full-diversity.

Lemma 1. *A lattice Λ is full-diversity under ML decoding on an L-diversity block-fading channel if and only if $\Psi_k x \in \mathbb{Z}^n$ admits $x = 0 \in \mathbb{R}^{\kappa(k)}$ as unique solution, $\forall k = 1 \ldots 2^L - 2$.*

The proof for Lemma 1 is based on a union bound on the pairwise error probability $P(0 \to x)$ and omitted due to lack of space.

Notice that Lemma 1 does not need H to be a low-density matrix. It is valid for both sparse and non-sparse matrices. It is possible to simplify Lemma 1 by reducing the number of constraints to L instead of $2^L - 2$, as stated in Lemma 2. Nevertheless, we believe that Lemma 1 is more useful for lattice construction than Lemma 2 because we usually start constructing a lattice in a recursive way by imposing lower diversity orders before reaching full-diversity. Consider the L largest Ψ_k integer-check matrices, those whose size is $n \times (n-$

n/L) and index is $k = 2^L - L - 1 \ldots 2^L - 2$. Let us refer to these matrices by $\Theta_k = \Psi_{k+2^L-L-2}$, for $k = 1 \ldots L$.

Lemma 2. *A lattice Λ is full-diversity under ML decoding on an L-diversity block-fading channel if and only if $\Theta_k x \in \mathbb{Z}^n$ admits $x = 0 \in \mathbb{R}^{n-n/L}$ as the unique solution, $\forall k = 1 \ldots L$.*

Proof for Lemma 2 is based on Lemma 1 and omitted due to lack of space.

As a direct application, Lemma 1 is followed by Theorem 1 stating how to construct a full-diversity lattice under ML decoding for $L = 2$. It is straighforward to generalize the proposed construction to $L > 2$. In the rest of this paper, we shall restrict the study to $L = 2$.

Theorem 1. *Consider a double-diversity block-fading channel. Let $H{=}[h_{ij}]$ be the $n \times n$ integer-check matrix of a real lattice Λ of even rank n, where $h_{ij} \in \mathbb{Q}$, the field of rationals. Let us decompose H into four $n/2 \times n/2$ submatrices as follows*

$$H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Assume that A, B, C, and D have full rank.
Let θ_1 and θ_2 be two algebraic numbers of degree ≥ 2 such that $\theta_2/\theta_1 \notin \mathbb{Q}$. Then, the two lattices defined respectively by the integer-check matrices

$$\begin{bmatrix} \theta_1 A & \theta_1 B \\ \theta_2 C & \theta_2 D \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \theta_1 A & \theta_2 B \\ \theta_2 C & \theta_1 D \end{bmatrix} \tag{10}$$

are full-diversity lattices under ML decoding.

Proof: Consider the first constraint $\Psi_1 x \in \mathbb{Z}^n$, where $x \in \mathbb{R}^{n/2}$ and

$$\Psi_1 = \begin{bmatrix} \theta_1 A \\ \theta_2 C \end{bmatrix}.$$

The upper half is $\theta_1 A x \in \mathbb{Z}^{n/2}$, we get $x \in \theta_1^{-1}\mathbb{Q}^{n/2}$. Similarly, The lower half is $\theta_2 C x \in \mathbb{Z}^{n/2}$, we get $x \in \theta_2^{-1}\mathbb{Q}^{n/2}$. But since $\theta_1\mathbb{Q} \cap \theta_2\mathbb{Q} = \{0\}$ we obtain $x = 0$.
A similar reasoning can be made for Ψ_2. Then, applying Lemma 1 yields the full-diversity construction. ■

The weak condition $\theta_2/\theta_1 \notin \mathbb{Q}$ enables us to use conjugate algebraic numbers from the same number field, e.g., take $\theta_1 = \frac{1+\sqrt{5}}{2}$ and $\theta_2 = \frac{1-\sqrt{5}}{2}$ in $\mathbb{Q}(\sqrt{5})$. A stronger condition may be defined as $\mathbb{Q}(\theta_1) \cap \mathbb{Q}(\theta_2) = \mathbb{Q}$ and could be benefitial for the coding gain but it is not required for full-diversity. Furthermore, as shown in the next section, the same construction combining H, θ_1, and θ_2, leads to a full-diversity lattice under iterative belief propagation decoding. A supplementary condition on the binary image of H is required to accomplish full-diversity with iterative decoding.

IV. Full-Diversity Construction of LDLC under Iterative Decoding

We keep restricting the study to the default diversity order $L = 2$ unless otherwise stated. Hereafter, we consider only real LDLC under iterative decoding, i.e., real lattices Λ with a sparse $n \times n$ integer-check matrix H. The lattice constraint $Hx \in \mathbb{Z}^n$ admits a bipartite graph representation as follows: (i) Draw n vertices (variable nodes) on the left representing the n lattice components x_j, $j = 1 \ldots n$. (ii) Draw n verticles (check nodes) on the right representing the n rows h_i of H that define the n LDLC constraints $h_i \cdot x = \sum_{j=1}^{n} h_{ij}x_j \in \mathbb{Z}$, $i = 1 \ldots n$. (iii) Link x_j and h_i by an edge if $h_{ij} \neq 0$. That edge has a multiplicative weight h_{ij}.

The factor graph ([10], chapter 2) defined above is used for iterative belief propagation of Λ [4]. Usually, H is regular with d non-zero entries per row and d non-zero entries per column, $d << n$. Let H_b be the incidence matrix of the factor graph, i.e., $H_b = [b_{ij}]$ where $b_{ij} = 1$ if $h_{ij} \neq 0$ otherwise $b_{ij} = 0$.

Definition 3. *The binary image $\mathcal{C}(\Lambda)$ of Λ is a binary LDPC code defined by its integer-check matrix H_b.*

As a direct consequence, the binary image $\mathcal{C}(\Lambda)$ has dimension 0 (0-rate) and length n. In general, for regular and irregular LDLC, the degree distribution of the binary image from an edge perspective [10] is given by $\lambda(x) = \sum_i \lambda_i x^{i-1}$ on the left and $\rho(x) = \sum_j \rho_j x^j$ on the left, where $\lambda(1) = \rho(1) = 1$ and $\sum_i \frac{\lambda_i}{i} = \sum_j \frac{\rho_j}{j}$.

Definition 4. *We say that Λ satisfies the Erasure Channel (EC) condition if the LDPC code $\mathcal{C}(\Lambda)$ achieves full-diversity [11] after a finite or an infinite number of decoding iterations.*

The EC condition is a necessary condition (but not sufficient) for Λ to achieve full-diversity. In a way similar to the study of full-diversity LDPC codes, the so-called *root-LDPC* [12][11], we redefine full-diversity under iterative belief propagation. The symbol error probability referred to as P_{es} is the error probability per lattice component:

Definition 5. *Consider a fading channel with L independent fading coefficients per lattice point. Λ is a full-diversity lattice under iterative decoding if the symbol error probability P_{es} at the iterative probabilistic decoder output is proportional to $\frac{1}{\gamma^L}$, for $\gamma = \frac{1}{\sigma^2} >> 1$.*

Before analyzing the construction of Theorem 1 under iterative decoding, let us take a look at LDLC lattices with a random structure. For random lattices and asymptotically large n, $C(\Lambda)$ is an ensemble of 0-rate binary LDPC codes with left and rigt degree distributions defined by the polynomials $\lambda(x)$ and $\rho(x)$ respectively. If the degree distribution is well chosen, a 0-rate ensemble can achieve the capacity of an ergodic binary erasure channel (BEC) with erasure probability ϵ_0, for any $\epsilon_0 < 1$. When Λ is transmitted on a block-fading channel with diversity order L, the random 0-rate LDPC ensemble $\mathcal{C}(\Lambda)$ will observe an ergodic binary erasure channel whose parameter is

$$\epsilon_0 = \frac{n - n/L}{n} = 1 - \frac{1}{L}. \tag{11}$$

This value of ϵ_0 is in accordance with the size of the largest integer-check matrices Θ_k used in Lemma 2 under ML decoding.

The diversity population evolution (DPE) tracks the fraction of full-diversity bits with the number of decoding iterations. The DPE renders a standard Density Evolution (DE) on the BEC, where $\epsilon_0 = 1 - \frac{1}{L}$, and

$$\epsilon_{i+1} = \left(1 - \frac{1}{L}\right) \lambda\big(1 - \rho(1 - \epsilon_i)\big). \tag{12}$$

The necessary condition EC for full-diversity is achieved if $\epsilon_i \to 0$ when $i \to +\infty$, i being the decoding iteration number. From (12) it is easy to prove the following propositions:

Proposition 1. *Consider a regular random LDLC ensemble with degree $d \geq 2$. For $L = 2$, the EC condition for full-diversity is satisfied iff $d \leq 7$.*

The above proposition tells us that the diversity tunnel is open for all regular random LDLC ensembles when $2 \leq d \leq 7$. The tunnel is closed for $d \geq 8$.

Proposition 2. *Consider a regular random LDLC ensemble with degree $d = 3$. The EC condition for full-diversity is satisfied iff $2 \leq L \leq 6$.*

The above proposition tells us that a 3-regular random LDLC can never be full-diversity for $L \geq 7$.

Irregular random LDLC may be useful to increase the fraction of full-diversity lattice components at the first decoding iterations and to increase the upper limit for achievable d and L. As an example, $\lambda(x) = 0.418683 \cdot x + 0.162635 \cdot x^2 + 0.418683 \cdot x^5$ and $\rho(x) = x^2$ has a better DPE tunnel than the fully 3-regular case.

Now, we can state an equivalent to Theorem 1 in the iterative decoding context. The construction mentioned here is given for $L = 2$ and any average weight $d \geq 2$.

Theorem 2. *Consider a double-diversity block-fading channel. Let H=$[h_{ij}]$ be the $n \times n$ integer-check matrix of a real lattice Λ of even rank n, where $h_{ij} \in \mathbb{Q}$, the field of rationals. Let us decompose H into four $n/2 \times n/2$ submatrices as follows*

$$H = \left[\begin{array}{cc} A & B \\ C & D \end{array} \right].$$

Assume that the binary image of H has the following structure:

$$H_b = \left[\begin{array}{cc|cc} \Pi_1 & 0 & B_2 & \Pi_4 \\ B_1 & \Pi_2 & \Pi_3 & 0 \\ \hline 0 & \Pi_6 & \Pi_7 & B_4 \\ \Pi_5 & B_3 & 0 & \Pi_8 \end{array} \right].$$

where Π_i are permutation matrices and B_i are regular random matrices with weight $d-2$.
Let θ_1 and θ_2 be two algebraic numbers of degree ≥ 2 such that $\theta_2/\theta_1 \notin \mathbb{Q}$. Then, the two lattices defined respectively by the integer-check matrices

$$\left[\begin{array}{cc} \theta_1 A & \theta_1 B \\ \theta_2 C & \theta_2 D \end{array} \right] \quad \text{and} \quad \left[\begin{array}{cc} \theta_1 A & \theta_2 B \\ \theta_2 C & \theta_1 D \end{array} \right] \tag{13}$$

are full-diversity lattices under iterative probabilistic decoding.

Proof: Let us assume that the lattice Λ is constructed according to one of the matrix given in (13) and $d = 3$. Since the error probability of the iterative decoder for LDLC is independent of the transmitted codeword [4], we can further assume that some random point $\boldsymbol{x} = (x_1, \ldots, x_n)$ was transmitted. Let us select check node i which is connected to variable nodes j_1, j_2, j_3.

The integer-check equation for the check node j for the point $\boldsymbol{x}$ can be written as follows

$$h_{i,j_1} x_{j_1} + h_{i,j_2} x_{j_2} + h_{i,j_3} x_{j_3} = z \ , \tag{14}$$

where $z \in \mathbb{Z}$. For ease of exposition, the subscript i is dropped and j_1, j_2, j_3 is replaced with $1, 2, 3$, respectively. If (14) is divided with h_{j_3} then it can be written as

$$x_3 = \tilde{h}_1 x_1 + \tilde{h}_2 x_2 + \tilde{h}_3 z \ , \tag{15}$$

where $\tilde{h}_1 = \frac{-h_1}{h_3}, \tilde{h}_2 = \frac{-h_2}{h_3}$ and $\tilde{h}_3 = \frac{1}{h_3}$.

Point $\boldsymbol{x}$ is transmitted over a BF channel with coefficients α_1 and α_2. The noise variance satisfies$\sigma^2 < \sigma^2_{\max}(\alpha_1, \alpha_2)$ where $\sigma^2_{\max}(\alpha_1, \alpha_2)$ is calculated according to (5). The construction proposed in (13) enforces that for a given integer-check j exactly one component out of x_1, x_2, x_3 would be affected by one of the channel coefficient whereas remaining two components would be affected by the other channel coefficient. With this lets assume that the components x_1, x_2 are affected by α_1 and x_3 is affected by α_2. Then the components for the received point $\boldsymbol{y}$ can be given by

$$y_1 = \alpha_1 x_1 + \eta_1, \ y_2 = \alpha_1 x_2 + \eta_2, \ y_3 = \alpha_2 x_3 + \eta_3, \tag{16}$$

where $\eta_k \sim \mathcal{N}(0, \sigma^2), k \in \{1, 2, 3\}$ is the noise.

The iterative decoder for LDLC initialize the variable nodes with probability density function (pdf) calculated from components of the received point. The pdf for components x_1, x_2, x_3 can be calculated as follows from (16),

$$f_i(t) \sim \mathcal{N}\left(\frac{y_i}{\alpha_1}, \frac{\sigma^2}{\alpha_1^2}\right), \ i = 1, 2; \quad f_3(t) \sim \mathcal{N}\left(\frac{y_3}{\alpha_2}, \frac{\sigma^2}{\alpha_2^2}\right).$$

During the first half of the first iteration, the check node i sends message pdf $p_{12}(t)$ to variable node x_3. Assume $z = 0$ in (15), we get partial check node message $\tilde{p}_{12}(t)$ from which $p_{12}(t)$ can be derived. The $\tilde{p}_{12}$ is calculated from (15),

$$\tilde{p}_{12}(t) \propto f_1\left(\frac{t}{\tilde{h}_1}\right) * f_2\left(\frac{t}{\tilde{h}_2}\right)$$

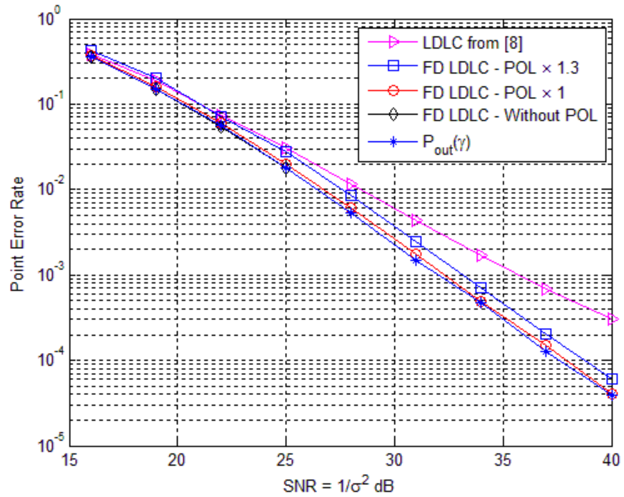
Here $*$ denotes the convolution operation. The convolution of n Gaussians with mean values $m_1, \ldots, m_n$ and variances $\sigma_1, \ldots, \sigma_n$, respectively, is a Gaussian with mean $m_1 + \ldots + m_n$ and variance $\sigma_1 + \ldots + \sigma_n$ [13][4]. Hence,

$$\tilde{p}_{12}(t) \sim \mathcal{N}\left(\frac{y_1 \tilde{h}_1 + y_2 \tilde{h}_2}{\alpha_1}, \ \frac{\left(\tilde{h}_1^2 + \tilde{h}_2^2\right)\sigma^2}{\alpha_1^2}\right). \tag{17}$$

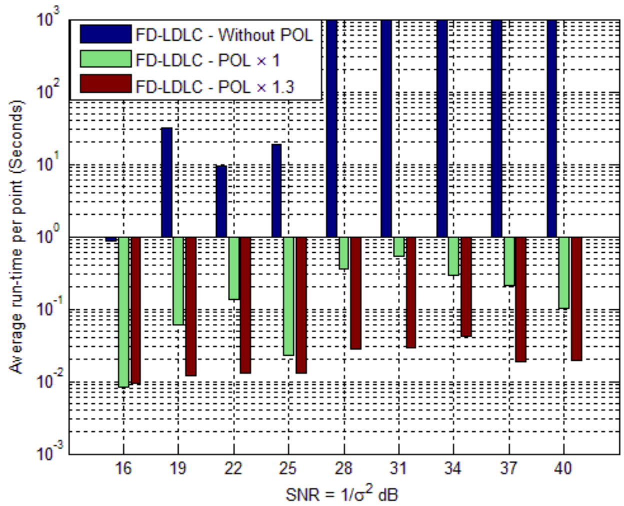
Now p_{12} can be derived from (17) as follows [4],

$$p_{12}(t) = \sum_{z=-\infty}^{\infty} \tilde{p}_{12}\left(t - \tilde{h}_3 z\right). \tag{18}$$

In the second half of the first iteration, the variable node x_3 multiplies channel pdf $f_3(x_3)$ and check node message p_{12} to generate updated pdf q_3. The multiplication of Gaussian mixtures also gives a Gaussian mixture. It is possible that the multiplication of the Gaussians at variable node generates a mixture of Gaussians with multiple peaks because p_{12} contain multiple peaks. However, here we assume that only the largest peak is retained from this mixture whereas other peaks are attenuated to zero. Due to this operation the resulting decoder is sub-optimal iterative decoder. Exact calculation of mean and variance of multiplication of the Gaussians can be found in [4].



(a) Point Error Rate Performance without POL, with POL defined by $\prod_{l=1}^{L}\alpha_l^2 = \frac{(2\pi e)^L}{\gamma^L}$, and with POL+1.3 margin defined by $\prod_{l=1}^{L}\alpha_l^2 = 1.3 \times \frac{(2\pi e)^L}{\gamma^L}$.



(b) Runtime comparison at different values of signal-to-noise ratio. A huge POL gain in runtime is observed at $n = 64$.

Fig. 1: ML decoding of double-diversity LDLC with dimension $n = 64$.

Lets assume that after multiplication and removal of smaller peaks, the only remaining peak in q_3 corresponds to the peak in p_{12} for which $z = 0$. Then,

$$q_3(t) \sim \mathcal{N}\left(m_3, \sigma_3^2\right), \tag{19}$$

where

$$m_3 = \frac{((y_1\tilde{h}_1 + y_2\tilde{h}_2)\alpha_1) + (y_3\alpha_2)(\tilde{h}_1^2 + \tilde{h}_2^2)}{\alpha_1^2 + \alpha_2^2(\tilde{h}_1^2 + \tilde{h}_2^2)}, \tag{20}$$

and σ_3^2 is the variance of q_3. Exact equation for σ_3^2 is omitted here. From (20),

$$\begin{aligned} m_3 &\propto (\tilde{h}_1 y_1 + \tilde{h}_2 y_2)\alpha_1 + (\tilde{h}_1^2 + \tilde{h}_2^2)\alpha_2 y_3 \\ &= (\alpha_1 x_3 + \tilde{h}_1\eta_1 + \tilde{h}_2\eta_2)\alpha_1 + (\tilde{h}_1^2 + \tilde{h}_2^2)\alpha_2(\alpha_2 x_3 + \eta_3). \end{aligned}$$

Therefore,

$$m_3 \propto (\alpha_1^2 + \omega_1\alpha_2^2)x_3 + \eta'. \tag{21}$$

where $\omega_1 = \tilde{h}_1^2 + \tilde{h}_2^2$ and $\eta' = (\tilde{h}_1\eta_1 + \tilde{h}_2\eta_2)\alpha_1 + (\tilde{h}_1^2 + \tilde{h}_2^2)\alpha_2\eta_3$. We would like to decide x_3 from m_3 given in (21) but this equation informs us that this decision has error rate behaving like $1/\gamma^2$ (See Proakis [1]) because of the generalized χ^2 distribution of order 4 in $(\alpha_1^2 + \omega_1\alpha_2^2)$. The 4-th order χ^2 distribution guarantees the double diversity for the sub-optimal iterative decoder. Since the sub-optimal iterative decoder achieves full-diversity, the original iterative decoder is also guaranteed to achieve it.

The above analysis assume $z = 0$ however, it remains valid also for $z \in \mathbb{Z}\setminus\{0\}$ as for any other value of z only ω_1 and/or η' in (21) would be affected and χ^2 distribution of order 4 appearing in (21) would remain intact. ∎

The proposed construction method in Theorem 2 can be extended to arbitrary values of channel diversity L. Other full-diversity constructions may also exist but we described one of the simplest method in Theorem 2.

V. Simulation Results

In this section we report computer simulation results for the LDLC codes contructed according to Theorem 1 and Theorem 2. We consider a channel with diversity order $L = 2$ in our simulations.

As mentioned in Sec. II, for a given SNR value γ, the Poltyrev outage limit (POL) gives a upper bound on the σ^2 and consequently a lower bound on the value of $\prod_{l=1}^{L}\alpha_l$. If the value of $\prod_{l=1}^{L}\alpha_l$ is below this limit (i.e., an inadmissible fading channel states) then even an optimal ML decoder would almost surely make a decoding error. Hence, it is possible for ML decoder to output an error without decoding when channel is in inadmissible fading state. This fact is utilized here to speed up the ML decoder of full-diversity LDLC. In all simulations, we count up to 400 erroneous points for each SNR value.

Under ML decoding, we utilize the integer-check matrix with dimension $n = 64$ constructed according to the second matrix of (10) where we select $\theta_1 = 1$ and $\theta_2 = \sqrt{2}$. We do not use any shaping region for the selected LDLC and decode using the ML decoder proposed in [14]. We use a PC with Intel Xeon E5-2687W CPU clocked at 3.10 GHz. Along with point error rate (PER) results for these LDLCs, we also report results for total run time required to complete simulations.

Simulation results for the aforementioned LDLC are shown in Fig. 1. In our simulations, we compare the point error rate (PER) and simulation runtime for programs that do not utilize POL with those where POL is utilized to declare error without decoding. We report four results using the sphere decoder [14] for ML decoding : 1) Random LDLC from [4]; 2) FD-LDLC, POL is not used 3) FD-LDLC, POL is used to detect inadmissible fading channel states ; and 4) FD-LDLC, POL is multiplied by a constant and the new value is used to detect inadmissible fading channel states. $P_{\text{out}}(\gamma)$ is the POL for $L = 2$. The curve for PER is parallel to that of POL and

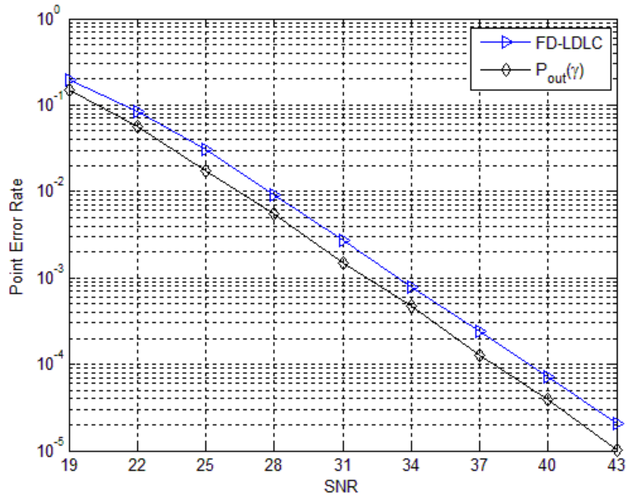


Fig. 2: Iterative decoding of double-diversity LDLC with dimension $n = 100$.

just 0.1dB away from it.

When random LDLC is used over BF channel, the PER for such LDLC is not proportional to $\frac{1}{\gamma^2}$ at high SNR values and this fact can be observed from Fig. 1(a).

We compare average runtimes required to decode a point in Fig. 1(b). It can be observed that when either the POL (case 3) or a scaled POL (case 4) is used for simulations, we get significant improvements in terms of runtime. The impact on runtime of ML decoding which utilizes POL is clearly visible for the selected LDLC. For this LDLC, it is not possible to simulate the PER performance for SNR values higher than 25dB without using POL in a feasible amount of time. The blue lines for SNR range 28dB to 40dB in Fig. 1(b) are saturated to the maximum possible runtime and are shown for reference only. Also for SNR values less than 25dB, the runtime with POL is only 10% of the runtime required without using POL. The difference in runtime between case 3 (POL only) and case 4 (scaled POL) for this particular LDLC is also very high.

For simulation using iterative decoder, we utilize the integer-check matrix with dimension $n = 100$ constructed according to the first matrix of (13) where we select $\theta_1 = 1$ and $\theta_2 = \frac{1}{\sqrt{2}}$. Again we do not use any shaping region. We use the iterative decoder for LDLC proposed in [4].Simulation result for this LDLC is given in Fig. 2. The curve for PER is parallel to that of POL ($P_{\text{out}}(\gamma)$) and hence this code exhibit diversity order of 2. In this case, the PER is around 1.5dB away from $P_{\text{out}}(\gamma)$. However, we expect the PER to approach POL as dimension n increases.

VI. Conclusions and Perspectives

We proposed construction methods for full-diversity lattices based on the integer-check matrix, the inverse of the lattice generator matrix. The first construction method is valid under ML decoding for both sparse and non-sparse integer-check matrices. The second construction method for full-diversity LDLC is based on sparse integer-check matrices and is valid for iterative decoding. Furthermore, we proved that lattice codes constructed according to the proposed methods do achieve full-diversity. We also verified the full diversity of our low-density lattices using computer simulations.

Theorem 1 is easily extendable to any diversity order L greater than 2 since it is based on two Lemmas with an arbitrary L. The extension of Theorem 2 to $L \geq 2$ requires an adequate choice of the binary image matrix H_b. Finally, the work in this paper focuses on the diversity only. Future work should take into account the coding gain of low-density lattices. For example, for a given L, find the matrix weight d that maximizes coding gain under ML and iterative decoding.

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