Non-binary adaptive LDPC codes for frequency selective channels: code construction and iterative decoding

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***Abstract*— We propose a new type of non-binary LDPC codes defined over multiple Galois fields. The so called multi-Galois LDPC code can adapt to the profile of a frequency selective channel and is suitable for multi-carrier transmissions with quadrature amplitude modulations.**

I. INTRODUCTION

The era of modern coding theory is mainly characterized by the development of powerful binary error-correcting codes that can be iteratively decoded. Robert Gallager proposed low-density parity-check (LDPC) codes with arbitrary alphabet size (see chap. 5 in [13]) a few years after the invention of the most famous non-binary error-correcting codes, i.e. the Reed-Solomon codes [6][21]. Following the empirical results by Davey and MacKay [8][9] on low-density parity-check codes built on finite fields alphabets, recent works appeared on the analysis and design of non-binary LDPC codes on both binary and non-binary channels. Authors have been motivated by the excellent results on the performance of LDPC codes [16][17]. The application and analysis of q-ary LDPC codes have been made by Bennatan and Burshtein under maximum likelihood decoding [3] and iterative decoding including the development of extrinsic information transfer charts [4] for arbitrary channels. Analysis via the Gaussian approximation has been proposed by Li et al. [19]. The density evolution of non-binary LDPC codes has been also studied for the erasure channel [25]. New constructions have been proposed for quasi-cyclic codes [20] and for non-binary codes via their binary image [24]. Finally, a very recent work by Kelley et al. [18] considers the pseudo-codewords of non-binary LDPC codes with two-dimensional modulations.

High data rate digital communication channels are non-binary. Symbols from non-binary LDPC codes can be naturally mapped into modulation symbols. Matching the alphabet of the error-correcting code to the channel alphabet explains the recent fever described above on studying non-binary LDPC codes. Similar examples can be found in the literature of communications and coding [26][14] before the rush on non-binary LDPC coding. An extremal case of code-channel matching is the example of orthogonal signals that can achieve the capacity of a Gaussian channel under coherent detection [30][29]. The efficiency of trellis coded modulations [28] is also due to code-modulation matching.

Let us look to matching or unmatching the code-modulation pair from a probabilistic decoding point of view. When a binary LDPC code is associated to a non-binary memoryless modulation, conversion of modulation symbol likelihoods into code bit probabilities is needed before decoding. In some cases, a backward conversion is also required where binary extrinsic probabilities generated by the LDPC decoder are fed back to the demodulator in order to close the iterative detection/decoding loop. Standard information theoretical tools [7] show that a significant loss in performance may be caused by symbol-to-bit and bit-to-symbol conversions. On the other hand, when the code alphabet is matched to the modulation alphabet, the channel likelihoods are directly processed by the decoder without any information loss, and there is no need to iterate between demodulation and decoding.

In this paper, we propose a non-binary LDPC code construction with multiple Galois fields. The Tanner graph of the code has symbol nodes belonging to different finite fields. The checknodes of our LDPC code are also connected to symbol nodes from various Galois fields. This LDPC code will be called *multi-Galois LDPC code*. In the context of frequency selective channels, this code is equivalently called *adaptive LDPC code* and will be denoted by C_a. We restrict the construction to simple checknodes defined by non-binary single parity-check (SPC) codes. For practical reasons, only the following finite fields of characteristic 2 are considered: $GF(2)$, $GF(4)$, $GF(8)$, $GF(16)$, $GF(64)$, and $GF(256)$. The LDPC code C_a may make use of all these fields at a time. The elements of $GF(2^m)$ will be matched to a complex symbol belonging to a bidimensional quadrature amplitude modulation (QAM) of size 2^m. The structure of C_a depends on the channel profile as described in the next section. In section III we give the generalized version of Hartmann-Rudolph decoding rule valid for any Galois field. This rule is then simplified for non-binary SPC codes, i.e. the checknodes of C_a. Experimental results on a multicarrier QAM-modulated selective channel are included in the last section of this paper.

II. MULTI-GALOIS CODE CONSTRUCTION

Let $\aleph_3$ denote the following ensemble of 4 nested finite fields starting from the smallest one and continuing by those having the order-3 element

$$\aleph_3 = \{GF(2), GF(4), GF(16), GF(256)\} \tag{1}$$

Similarly, let $\aleph_7$ denote the following ensemble of 3 nested finite fields starting from the smallest one and continuing by those having the order-7 element

$$\aleph_7 = \{GF(2), GF(8), GF(64)\} \quad (2)$$

The bipartite graph representation of C_a is given in Fig. 1. As usual, symbol nodes are drawn as circles and checknodes are drawn as rectangles. Edges from type-1 checknodes can only connect to symbols from $\aleph_3$. Edges from type-2 checknodes can only connect to symbol nodes in $\aleph_7$. The gluing of the two subgraphs is guaranteed by binary symbol nodes and by binary checknodes. The latter are of both type-1 and type-2. Binary symbol nodes are sometimes called *state variables* and their value may not be transmitted on the channel, i.e. state variables can be punctured.

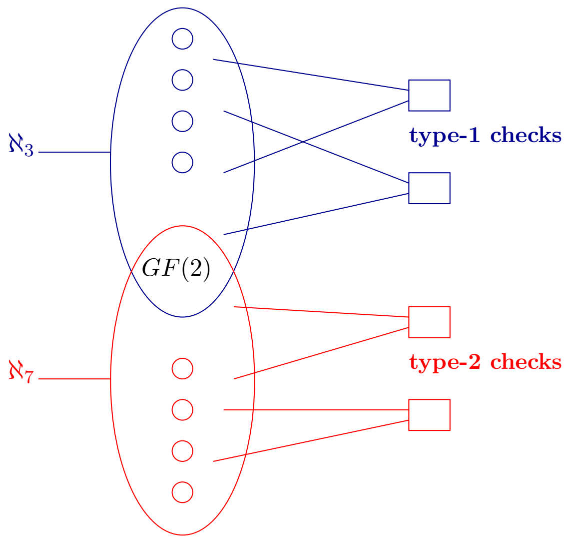

Fig. 1. Tanner graph representation of the multi-Galois LDPC code.

A checknode in C_a is a non-binary SPC code defined over $GF(Q)$. For type-1 checknodes, $GF(Q)$ must be in $\aleph_3$. Any symbol connected to type-1 checknodes belongs to $GF(q) \in \aleph_3$ where $2 \le q \le Q \le 256$. Similarly, for type-2 checknodes, $GF(Q)$ must be in $\aleph_7$. Any symbol connected to type-2 checknodes belongs to $GF(q) \in \aleph_7$ where $2 \le q \le Q \le 64$. While performing the sum-product probabilistic decoding of the multi-Galois code, a symbol node defined over $GF(q)$ sends a message on each outgoing edge represented by a vector of q extrinsic probabilities. If a checknode is defined over $GF(Q = q)$, it also receives a message represented by a vector of q probabilities on each of its incoming edges. In the case where $q < Q$, the symbol node must extend its message by zero padding.

Let N denote the number of symbol nodes in the bipartite graph, i.e. N is the non-binary length of C_a. Let N_q denote the number of symbols belonging to $GF(q)$, $q = 2^i$. Then, the binary image of C_a has length (expressed in bits)

$$N_b = \sum_{i=1,2,3,4,6,8} i \times N_{2^i} \quad (3)$$

On the right-hand of the Tanner graph, let L_q denote the number of checknodes defined over $GF(q)$. The number of equivalent binary checks is

$$L_b = \sum_{i=1,2,3,4,6,8} i \times L_{2^i} \quad (4)$$

If the state variables are not punctured, the adaptive multi-Galois code C_a has rate $R_c = K_b/N_b$ where $K_b = N_b - L_b$. The design rules described above can be briefly stated in terms of the parity-check matrix H_a defining C_a:

- Consider a column of H_a. All entries must belong to the same finite field.
- Consider a row of H_a. Entries can be selected from different finite fields, but they must not mix $\aleph_3$ and $\aleph_7$ except for binary entries.

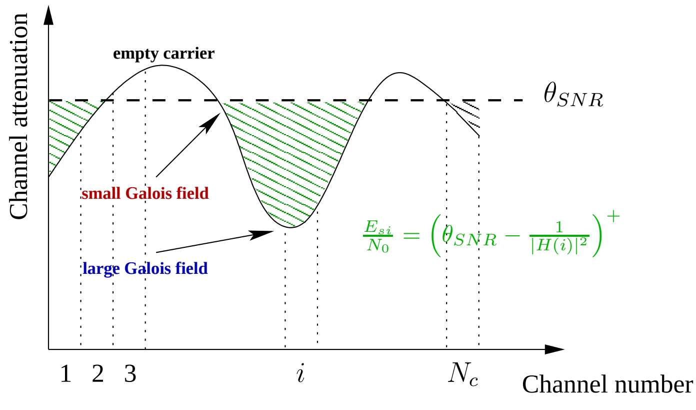

Fig. 2. Illustration of waterfilling associated with the multi-Galois code.

Now, let us describe how the parameters of C_a depend on the channel profile. The channel is assumed to be frequency selective (inter-symbol interference channel). A multi-carrier orthogonal frequency division multiplexing (OFDM) converts the channel into N_c parallel flat sub-channels, where N_c is the number of OFDM sub-carriers. For simplicity reasons and without loss of generality, it can be assumed that the length of C_a is less than or equal to the number of sub-carriers, $N \le N_c$. If the channel model is defined by a tapped delay line of length $\nu + 1$, the channel frequency response is

$$H(f) = \sum_{k=0}^{\nu} h_k e^{-j2\pi f \Delta_k / W} \quad (5)$$

where h_k is the complex coefficient of the kth path, Δ_k is the time delay of the kth path, W is the frequency bandwidth of the OFDM modulated signal, and ν is the channel memory. Sub-carrier i is located on frequency $f_i = (i-1) \times W/N_c$, where $i = 1 \ldots N_c$. An illustration of the channel frequency profile is given in Fig. 2. The signal-to-noise ratio E_{si}/N_0 of channel i can be determined by the classical waterfilling technique [7]. The water level θ_{SNR} maximizes Shannon capacity under the constraint of a fixed total transmitted energy. Then, the capacity is evaluated by

$$C_i = \log_2 \left(1 + |H(f_i)|^2 \cdot \frac{E_{si}}{N_0} \right) \tag{6}$$

The 2^m-QAM modulation on channel i is chosen such that m is the maximum integer that satisfies the inequality $mR_c \leq C_i$. Other methods can be proposed for the adaptation of QAM size to the channel profile, we restrict ourselves to this simple method since QAM adaptation (also known as *bit allocation* in OFDM) is out of the scope of this paper.

Tap index	1	2	3	4	5	6	7	8	9	10	11	12
Delays	0	1	3	5	8	11	13	17	23	31	32	50
Powers (dB)	-4	-3	0	-2.6	-3	-5	-7	-5	-6.5	-8.6	-11	-10

TABLE I

URBAN AREA CHANNEL CHARACTERISTICS.

$R_c = 1/2$	N_2 2-QAM	N_4 4-QAM	N_8 8-QAM	N_{16} 16-QAM	N_{64} 64-QAM	N_{256} 256-QAM
SNR=7 dB	71	25	35	80	50	24

TABLE II

NUMBER OF SUB-CARRIERS FOR EACH MODULATION AT SNR = 7 DB.

L_2 2-QAM	L_4 4-QAM	L_8 8-QAM	L_{16} 16-QAM	L_{64} 64-QAM	L_{256} 256-QAM
0	16	24	54	34	14

TABLE III

NUMBER OF CHECKNODES FOR EACH GALOIS FIELD AT SNR=7DB.

Given the channel frequency response and the code length N, the number N_q of symbol nodes in $GF(q)$ is determined via (5) and (6). Then, by taking into account the constraints imposed by the design rules of C_a and the degree distribution of nodes in the graph, the number L_q of checknodes defined over $GF(q)$ is determined via integer linear programming.

As an example, a urban area mobile radio channel is given in Table I. The channel has 12 taps with delays multiple of 0.1 microseconds. For $N_c = 300$ sub-carriers, Table II indicates the number of sub-carriers for each modulation at SNR=7dB. Assuming that C_a is a regular (3,6) multi-Galois LDPC code, the number of checknodes in $GF(q)$ found by integer linear programming is given in Table III. The number of punctured state variables is 36. The coding rate is $R_c = 0.401$.

III. NON-BINARY HARTMANN-RUDOLPH DECODING RULE

A posteriori probability decoding of checknodes in a non-binary multi-Galois code can be performed exhaustively at the expense of a high complexity in $O(q^{d-1})$ where d is the checknode degree. Fast optimal decoders have been proposed [5][10][27][1] and they are all based on exploiting the duality property. The original idea of decoding any linear binary $(n,k)_2$ code via its dual code is found in [15] and [2]. The complexity degree 2^k is replaced by 2^{n-k}. This duality property is simply the result of Poisson summation formula as elegantly explained by Forney [12].

We describe below a generalization of Hartmann and Rudolph decoding rule for Galois fields of size $q = p^m$ with characteristic p. We follow a similar proof as in [15] but with any $m \geq 1$. The obtained decoding rule can be applied to checknodes even more universal than SPC codes, e.g. on generalized low density codes [23]. Low complexity sub-optimal decoding rules for non-binary SPC codes have been also considered in the literature, e.g. [11].

Firstly, let us define the trace of an element and two characters in finite fields [22][21]. The trace maps $\beta \in GF(p^m)$ into an element in the original field, $\tau : GF(p^m) \to GF(p)$. Many definitions are equivalent, let us take $\tau(\beta) = \beta_0$, when $\beta = \beta_{m-1}\alpha^{m-1} + \ldots + \beta_1\alpha + \beta_0$ where α is a primitive element of $GF(p^m)$. The first character we define is $\chi_\beta : GF(p^m) \to \mathbb{C}$, $\chi_\beta(\gamma) = \omega^{\tau(\beta\gamma)}$, where $\omega = e^{2\pi\frac{\sqrt{-1}}{p}}$. The second character is $\underline{\chi}_u : GF(p^m)^n \to \mathbb{C}$, defined by $\underline{\chi}_u(v) = \omega^{\tau(<u,v>)}$, where $< u, v >$ is the scalar product of the two vectors u and v.

The non-binary decoding rule is: Set $\hat{c}_i = a$, where $a \in GF(q = p^m)$ maximizes the expression:

$$A_i(a) = \sum_{\beta \in GF(q)} \chi_\beta(-a).$$
$$\sum_{j=1}^{q^{n-k}} \left[\prod_{l=0}^{n-1} \sum_{\gamma \in GF(q)} \chi_{(c'_{jl} - \beta\delta_{il})}(-\gamma) P(y_l|\gamma) \right] \tag{7}$$

where $c = (c_0, c_1, ..., c_{n-1})$ is a codeword of $C(n,k)$ defined over $GF(q)$, $c'_j = (c'_{j0}, c'_{j1}, ..., c'_{jn-1})$ the jth codeword of the dual code $C^\perp(n, n-k)$, $y = (y_0, y_1, ..., y_{n-1})$ is the received word, $\hat{c}_i$ is the estimate of c_i, β and $\gamma \in GF(q)$, and $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

Theorem: Decoding rule (7) maximizes the probability that $\hat{c}_i$ equals c_i.

Proof : Let us show that $P(c_i = a|y) = \lambda A_i(a)$, where λ is a positive constant. We write

$$P(c_i = a|y) = \sum_{c \in C, c_i = a} P(y|c).[P(c)/P(y)] \tag{8}$$

Assume in a first step that $P(c) = p^{-mk}$ (equiprobable codewords), then (8) becomes

$$P(c_i = a|y) = [p^{-mk}/P(y)] \sum_{c \in C} P(c|y)\delta_{0,c.e_i - a} \tag{9}$$

where $e_i = (\delta_{i0}, \delta_{i1}, ..., \delta i(n-1))$. Using Fourier transform we can write,

$$\delta_{0,c.e_i - a} = q^{-1}. \sum_{\beta \in GF(q)} \chi_\beta(c.e_i - a) \tag{10}$$

$$P(y|c) = q^{-n}.\sum_{u\in GF(q)^n} F(y,u)\underline{\chi_u}(c) \quad (11)$$

and

$$F(y,u) = \sum_{v\in GF(q)^n} P(y|v)\underline{\chi_u}(-v) \quad (12)$$

Substituting (10) and (11) in (9) yields

$$P(c_i = a|y) = [p^{-m(n+k+1)}/P(y)] \cdot \sum_{\beta\in GF(q)} \chi_\beta(-a). \sum_{u\in GF(q)^n} F(y,u)[\underline{\chi}_{(u+\beta.e_i)}(c)] \quad (13)$$

The orthogonality property of group characters makes (13) as

$$P(c_i = a|y) = [p^{-m(n+k+1)}/P(y)] \cdot \sum_{\beta\in GF(q)} \chi_\beta(-a). \sum_{c'\in C^\perp} F(y, c' - \beta.e_i) \quad (14)$$

By assuming that the channel and modulation are memoryless we obtain

$$F(y,u) = \sum_{v\in GF(q)^n} \prod_{l=0}^{n-1} P(y_l|v_l)\chi_{u_l}(-v_l) = \prod_{l=0}^{n-1} \sum_{\gamma\in GF(q)} P(y_l|\gamma)\chi_{u_l}(-\gamma) \quad (15)$$

Substituting (15) in (14) leads to

$$P(c_i = a|y) = [p^{-m(n+k+1)}/P(y)] \cdot A_i(a) \quad (16)$$

□

For the special case of non-binary SPC codes on $GF(2^m)$ (the dual is a repetition code), we have

$$A_i(a) \propto P(y_i|a) \cdot FHT\{\prod_{l=0,l\neq i}^{n-1} FHT[P(y_l|\gamma)]\} \quad (17)$$

where FHT is a fast Hadamard transform on $q = 2^m$ points. As stated in the proof of the above theorem, if codewords are not equiprobable (which is the case in a multi-Galois LDPC), then simply multiply the observations in (17) by their symbol a priori information.

IV. Experimental results

Let us start with a toy example illustrated in Fig. 3. It includes an elementary 4-ary code $C_4(N_4 = 3, L_4 = 2, K_4 = 1)$, an elementary 8-ary code $C_8(N_8 = 3, L_8 = 2, K_8 = 1)$, and two binary symbol nodes with one binary checknode ($N_2 = 2$ and $L_2 = 1$). The state variables are punctured. The upper part of C_a is essential in order to glue the quaternary and the octal codes. The elementary codes C_4 and C_8 are of equal rate $1/3$. For the adaptive multi-Galois code C_a:
$N_b = N_2 + 2 * N_4 + 3 * N_8 = 17$
$L_b = L_2 + 2 * L_4 + 3 * L_8 = 11$
$K_b = N_b - L_b = 6$

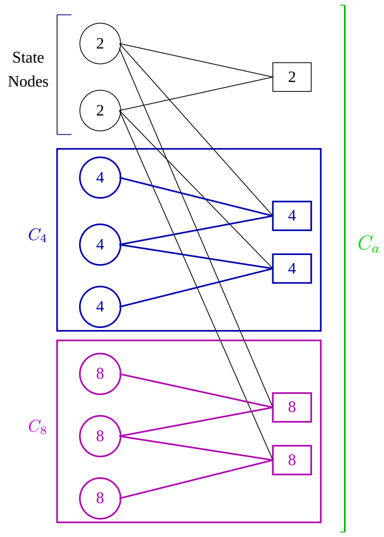

Fig. 3. Bipartite graph of a toy multi-Galois LDPC code.

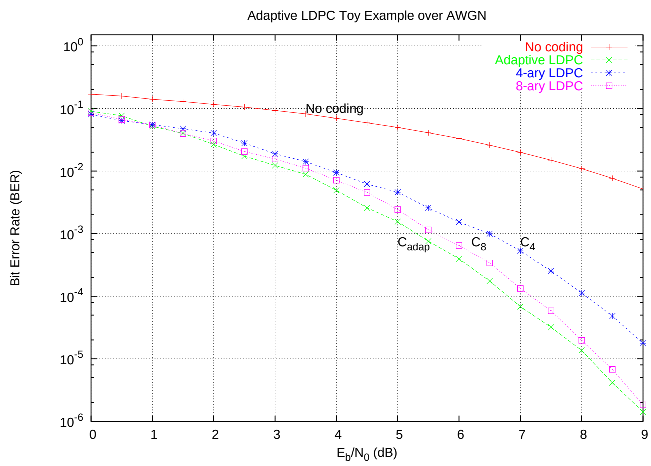

Fig. 4. Performance of the toy multi-Galois code defined in Fig. 3.

$R_c = K_b/(N_b - N_2) = 0.4$
The parity-check matrices of C_4, C_8, and C_a are respectively

$$H_4 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \quad H_8 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 5 & 6 \end{pmatrix}$$

$$H_a = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 5 & 6 \end{pmatrix}$$

The error rate performance of C_a is shown in Fig. 4 for an additive white Gaussian noise channel and a binary modulation. We easily remark that the adaptive code improve the performance of the two non-binary codes.
Figures 5 show the bit error rate for an adaptive multi-Galois LDPC code matched to an OFDM system with N_c=300 sub-carriers which are divided to N_2=75 and 5 equal number N_{2^i},

for i =2, 3, 4, 6 and 8. All variables are transmitted over the selective channel. The number of checknodes is L_2=0 and $L_q = 30$ for all non-binary checks. We have N_b=1110, L_b=690, and R_c=0.4. We compare this adaptive LDPC code to systems where we use a binary LDPC code and an adaptive modulation. The binary LDPC codes considered have coding rate equal to 0.5, 0.4, and 0.25. We remark in Fig. 5 that the adaptive LDPC code improves the performance by up to 1dB at BER=$10^{-}4$ with respect to the binary rate 0.4 code. It also approaches closely the one with rate 0.25. At high SNR, the behavior is dominated by the binary variable and binary check nodes. A similar behavior is observed on Fig. 6 for the Urban Area Channel given in Table I.

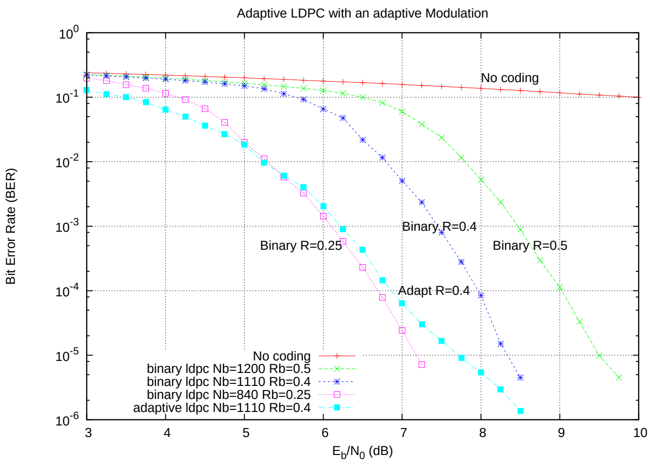

Fig. 5. Bit error rate for a multi-Galois LDPC code with OFDM.

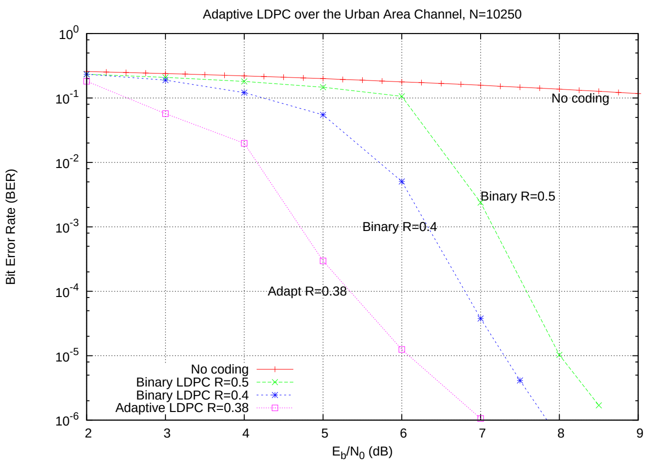

Fig. 6. Urban Area Channel, LDPC adapted at 7dB, N_b=10250 bits.

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