Exercice I - The Sphere Decoder, an example with $A_2$

Consider the hexagonal point lattice $A_2 \subset \mathbb{R}^2$. The studied lattice version is generated by the following generator matrix $M$ written in row convention

$$M = \begin{pmatrix} 1 & 0 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

A lattice point $x = (x_1, x_2) \in A_2$ is defined by $x = zM$, where $z = (z_1, z_2) \in \mathbb{Z}^2$. Hexagonal lattice points and their Voronoi cells are illustrated in Figure 1. Let $d_{E_{\text{min}}}$ denote the minimum Euclidean distance of $A_2$ generated by the above matrix $M$. The packing radius $\rho = d_{E_{\text{min}}}/2$ and the covering radius $R$ are depicted in Figure 2.

1) Determine $\rho$ and $R$.
2) What is the volume of a Voronoi cell?
3) Check that the packing density $\Delta$ of $A_2$ is equal to 0.906, i.e. 90.6% of the bidimensional space is covered by the packing balls of $A_2$.
4) Let $G = MM^t = [g_{ij}]$ be the Gram matrix. We define the following quadratic form $Q(z)$ associated to the squared Euclidean norm

$$Q(z) = \|x\|^2 = xx^t = zGz^t = \sum_i \sum_j g_{ij}z_iz_j$$

Prove that $Q(z)$ can be written as $Q(z) = q_{11}Z_1^2 + q_{22}Z_2^2$, where $Z_1 = z_1 + q_{12}z_2$ and $Z_2 = z_2$. Find the values of $q_{ij}$. For higher dimensions $n \geq 2$, a Cholesky or a QR decomposition is applied in order to write $Q(z) = \sum_{i=1}^n q_{ii}Z_i^2$.

5) The enumeration of lattice points satisfying $\|x\|^2 \leq C$, where $C \in \mathbb{R}^+$ is a fixed squared radius, is equivalent to solving $Q(z) \leq C$. Let $S(0, \sqrt{C})$ denote the sphere with center 0 and radius $\sqrt{C}$. Prove that enumerating lattice points inside $S(0, \sqrt{C})$ is equivalent to solving
the following recursive inequalities

\[-\sqrt{\frac{C}{q_{22}}} \leq z_2 \leq +\sqrt{\frac{C}{q_{22}}}\]

\[-\sqrt{\frac{C-q_{22}z_2^2}{q_{11}}} - q_{12}z_2 \leq z_1 \leq +\sqrt{\frac{C-q_{22}z_2^2}{q_{11}}} - q_{12}z_2\]

(3)

Fig. 1 – The hexagonal point lattice \(A_2\) in the bidimensional plane.

Fig. 2 – Representation of the packing radius \(\rho\) and the covering radius \(R\).

6) Let \(y = (y_1, y_2) \in \mathbb{R}^2\) be a randomly chosen point, e.g. \(y\) may be the output of a Gaussian channel with \(y = x + \eta\), where \(\eta\) is an additive white Gaussian noise and \(x \in A_2\). Let \(\epsilon\) be a
vanishing positive real number.

6.a) Prove that $S(y, R + \epsilon)$ contains at least one lattice point.
6.b) What is the maximum number of lattice points inside $S(y, R + \epsilon)$?

7) A sphere decoder is a decoder that finds the closest lattice point to $y$ by enumerating all lattice points inside $S(y, R + \epsilon)$ and keeping the nearest one. Hence, the sphere decoder for $A_2$ takes $C = R^2 + \epsilon$ and solves $\|y - x\|^2 \leq C$ in order to get

$$x_{ML} = \arg \min_{x \in A_2} \|y - x\|^2$$

7.a) Prove that the sphere decoder is given by the inequalities of question 5 above, where $z$ is replaced by $z - \xi$ and $\xi = yM^{-1}$.
7.b) Apply the sphere decoder in order to find the closest lattice point $x_{ML}$ to the received noisy point given by $y = (1.75, 1.75)$.

**Exercice II - The Gosset lattice $E_8$, complex construction**

The point lattice $E_8$ yields the densest lattice packing in $\mathbb{R}^8$. It is the unique lattice in dimension 8 with an Hermite constant (fundamental gain) equal to 2, and a kissing number equal to 240. Recall that the fundamental gain of a real lattice $\Lambda \in \mathbb{R}^n$ of rank $n$ is defined as

$$\gamma(\Lambda) = \frac{d^{2}_{E_{min}}(\Lambda)}{\sqrt[2n]{\text{vol}(\Lambda)}}$$

where $\text{vol}(\Lambda) = |\text{det}(M)|$, $M$ being a square $n \times n$ generator matrix of $\Lambda$. The kissing number $\tau$ is given by the number of lattice points on the first lattice shell, also equal to the number of neighbouring packing balls tangent to the one centered on the origin.

Let $\mathcal{G} = \mathbb{Z}[i] \sim \mathbb{Z}^2$ denote the ring of Gaussian integers. Let $\phi = 1 + i \in \mathcal{G}$, where $i = \sqrt{-1}$. The ring $\mathcal{G}$ can be partitioned via two subgroups, i.e. the partition chain $\mathcal{G}/\phi \mathcal{G}/\phi^2 \mathcal{G}$ is used. At depth 1, we have $\mathcal{G} = \phi \mathcal{G} + [\mathcal{G}/\phi \mathcal{G}]$. At depth 2, we have $\mathcal{G} = \phi^2 \mathcal{G} + [\phi \mathcal{G}/\phi^2 \mathcal{G}] + [\mathcal{G}/\phi \mathcal{G}]$.

1) Find the order of the quotient groups $[\mathcal{G}/\phi \mathcal{G}]$ and $[\phi \mathcal{G}/\phi^2 \mathcal{G}]$. Tip : Determine the fundamental volumes of both lattices $\mathcal{G}$ and $\phi \mathcal{G}$. Find also $[\mathcal{G}^N/g \mathcal{G}^N]$, where $g \in \mathcal{G}$ and $N \in \mathbb{N}$.
2) Give the typical coset leaders of the partition $\mathcal{G}/\phi \mathcal{G}$, i.e. the typical elements of the quotient group $[\mathcal{G}/\phi \mathcal{G}]$. Use Figure 3 for illustration.
3) Give the typical coset leaders of the partition $\phi \mathcal{G}/\phi^2 \mathcal{G}$, i.e. the typical elements of the quotient group $[\phi \mathcal{G}/\phi^2 \mathcal{G}]$. Use Figure 4 for illustration.
4) Consider the complex lattice \( \Lambda = (4, 1, 4) + \phi(4, 3, 2) + \phi^2 G^4 \), where \((4, 1, 4)\) is the binary repetition code of length 4, and \((4, 3, 2)\) is the binary single parity-check code of length 4. In the formula of \( \Lambda \), binary elements 0 and 1 should be embedded into the complex ring \( G \).

4.a) Consider the chain \( G^4/\Lambda/\phi^2 G^4 \). Deduce the fundamental volume of \( \Lambda \).

4.b) Determine the minimum Euclidean distance of \( \Lambda \) and hence its Hermite constant. At this point, you must find that \( \Lambda = E_8 \).

4.c) From the complex formula of \( E_8 \), check that \( \tau = 240 \) by enumerating all lattice points of shortest length.

5) An encoder for a finite size constellation carved from \( E_8 \) is shown in Figure 5. This encoder is based on the formula \( E_8 = (4, 1, 4) + \phi(4, 3, 2) + \phi^2 G^4 \) (B construction).

5.a) What is the number of uncoded bits per complex symbol in the coded 64-QAM? What should be the uncoded QAM reference constellation?

5.b) Compare the coded 64-QAM and the uncoded reference, and find again the 3dB gain over a Gaussian channel given in the Hermite constant.

![Diagram](image_url)

**Fig. 3** – Depth-1 partitioning of a 64-QAM corresponding to \( G = \phi G + [G/\phi G] \).
Fig. 4 – Depth-2 partitioning of a 64-QAM corresponding to $\mathcal{G} = \phi^2 \mathcal{G} + [\phi \mathcal{G}/\phi^2 \mathcal{G}] + [\mathcal{G}/\phi \mathcal{G}]$.

Fig. 5 – Encoder for an $E_8$ constellation. Information rate is 5 bits per complex dimension.