Information theoretical versus algebraic constructions of linear unitary precoders for non-ergodic multiple antenna channels

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***Abstract*— In this paper, we propose a new construction of linear unitary precoders for multiple antenna fading channels based on an information theoretical tool and we compare its performance with some known algebraic precoders. The new precoder has been selected in the ensemble of random unitary precoders by minimizing the outage probability associated to the instantaneous mutual information. Then we consider a bit-interleaved coded modulation where a non-systematic non-recursive convolutional code is cascaded with a linear modulation and a block-fading multiple antenna channel. The word error rate performance of the proposed precoder is illustrated under iterative detection and decoding.**

I. Introduction

The need to transmit at high data rates with better performance has recently motivated research on signal processing and coding for multiple-input multiple-output (MIMO) channels. As multiple antenna systems have large capacity, design criteria have been set up for space-time signals used in these systems through minimizing the pairwise word error probability [11][19]. Orthogonal and algebraic designs allowed to achieve very high data rates with full diversity and very good performance in uncoded systems and quasi-static fading environments [8][17][2].

Precoding signals for fading channels, which is well known in single antenna transmissions, has been rediscovered for multiple antenna channels. Battail was the first to suggest rotations to combat channel fluctuations [1]. The pioneering work on multi-dimensional rotated modulations achieved in the nineties, such as [3][7][4], opened the way for the study of multi-dimensional rotations (i.e. linear unitary precoders) in MIMO channels.

Rotations in single antenna systems have been designed by classical algebraic criteria, except for orthogonal transforms proposed by Rainish which are based on the minimization of the cut-off rate [16]. Also, it has been shown in [15] that random rotations perform as good as algebraic rotations in a high-diversity high-dimensional environment. In [13], the authors proposed an information theoretical tool to design space-time codes. They designed codes that maximize the ergodic capacity of the channel, but such space-time codes perform poorly on a non-ergodic block fading channel. Furthermore, space-time signal modulations must be combined to error-correcting codes in order to achieve optimal performance in the information theoretical sense.

In [5][10], the authors considered bit-interleaved coded modulations [6] for space-time coding (ST-BICM). They showed that quasi-optimal global ML performance of the coded modulation is achieved by imposing specific constraints (called *genie conditions*) on the structure of the space-time precoder. The ML performance is attained in practice after some iterations in a joint detection/decoding process at a high signal-to-noise ratio.

We propose in this article full-rate rotations that minimize the outage probability of a MIMO system in block fading environments. The paper is organized as follows: section II presents the system model and notations, section III describes different design techniques for space-time codes, including the outage minimization criterion. Computer simulation results and conclusions are drawn in the last section.

II. System model and notations

Consider a multiple-input multiple-output system with n_t transmit and n_r receive antennas. The channel is supposed to be frequency non-selective (no inter-symbol interference) and known to the receiver but not to the transmitter (CSI at the receiver side only). The number of independent channel realizations observed during one codeword transmission is denoted by n_c. The parameter n_c takes values from 1 (quasi-static fading) up to the codeword time length (ergodic channel). The input-output channel model is given by

$$y = zSH + \eta \tag{1}$$

where $z \in (2^m - \text{QAM})^{sn_t}$, S is the linear precoder matrix (also called full-rate space-time block code in the MIMO community) of size $sn_t \times sn_t$. The integer parameter s represents the time spreading of the precoder, $1 \leq s \leq n_t n_c$. We restrict our study to unitary precoders, i.e., $S^{-1} = S^h$. The $sn_t \times sn_r$ MIMO channel matrix H has s blocks on its diagonal and zeros elsewhere. Each block is associated to the transmission over the $n_t \times n_r$ MIMO channel during one channel use. We can see SH as a new correlated MIMO channel and call a precoding time period the group of s channel uses associated to the matrix SH. Finally, $y \in \mathbb{C}^{sn_r}$ is the channel output and η is an additive white gaussian noise with zero mean and variance σ^2 per real component.

III. Linear precoding designs

A. Known Design Methods

Let us briefly recall some of the known methods used to design space-time precoders.

• Design based on pairwise error probability: this is the well-known criterion of maximizing the rank and the determinant of the difference matrix [11][19]. Algebraic methods were used to design space-time codes that are full-rate [17][2][8], i.e. one symbol per transmit antenna per symbol time. Such algebraic space-time precoders have been optimized for uncoded linear modulations under maximum-likelihood detection. Some of them do not exist for all values of n_t. Furthermore, all precoders designed by this criterion for the quasi-static channel have been developed with full-spreading where $s = n_t$. As an example, the Golden code [2] is an algebraic precoder optimized for $n_t = n_r = s = 2$, its precoding matrix is

$$S = \begin{bmatrix} 0.52e^{-j0.55} & 0 & 0 & 0.85e^{+j1.01} \\ 0.85e^{-j0.55} & 0 & 0 & 0.52e^{-j2.12} \\ 0 & 0.85e^{+j2.58} & 0.52e^{-j0.55} & 0 \\ 0 & 0.52e^{-j0.55} & 0.85e^{-j0.55} & 0 \end{bmatrix} \quad (2)$$

The weak point in constructing rotations via the pairwise error probability criterion is the fact that error-correcting codes that may be serially concatenated with the precoder are not taken into account by the design criterion.

• Design maximizing the ergodic channel capacity: linear dispersion (LD) codes [13] are designed for multiple antenna channels by a search that maximizes the ergodic capacity under a gaussian channel input. Such a design is not necessarily suitable for a block fading channel with a finite number of states, e.g., $n_c = 1, 2, 3$, etc. Also, the kind of input alphabet is not considered in the search for linear dispersion codes.

• Design based on genie conditions: this construction was introduced in [5] where perfect *a priori* probability feedback is assumed in the iterative joint detection and decoding of ST-BICM. In order to guarantee maximum diversity order and maximum coding gain at the output of the detector, the design must guarantee two conditions: 1- Orthogonal sub-rows in the linear precoding matrix, and 2- Equal norm sub-rows in the linear precoding matrix. In ST-BICM, there exists a strong interaction between the error correcting code with interleaving and the linear precoder, both in terms of diversity and coding gain maximization [10]. Complexity can be controlled by the choice of a minimal spreading factor s that guarantees full diversity [9]. The genie conditions are optimal, in terms of ML performance, when all diversity given by the transmit antennas is collected at the detector (i.e. $s = n_t$). A supplementary condition called "Dispersive Nucleo Algebraic" (DNA) has been proposed in [10] to keep optimality when $s < n_t$ while having the genie conditions on sub-groups of transmit antennas.
As an example, the cyclotomic rotation given below is an algebraic precoder satisfying the genie conditions for ST-BICM with $n_t = s = 2$:

$$S_{Cyclo} = \frac{1}{2}\begin{bmatrix} 1 & 1 & e^{j6\pi/15} & -e^{j6\pi/15} \\ e^{j2\pi/15} & je^{j2\pi/15} & -e^{j8\pi/15} & je^{j8\pi/15} \\ e^{j4\pi/15} & -e^{j4\pi/15} & e^{j10\pi/15} & e^{j10\pi/15} \\ e^{j6\pi/15} & -je^{j6\pi/15} & -e^{j12\pi/15} & -je^{j12\pi/15} \end{bmatrix} \quad (3)$$

Consider the space-time bit-interleaved coded modulation drawn in Fig. 1. This ST-BICM is a serial concatenation of a rate R_c binary convolutional code, an interleaver of size N bits, and a QAM mapper followed by the precoder as described in the previous section. When S is the identity matrix, the ST-BICM diversity order is upper-bounded by [14]:

$$d \leq n_r \left(\lfloor n_c n_t (1 - R_c)\rfloor + 1\right) \quad (4)$$

The maximal diversity given by the outage limit under a finite size QAM alphabet also achieves the above Singleton bound [12].

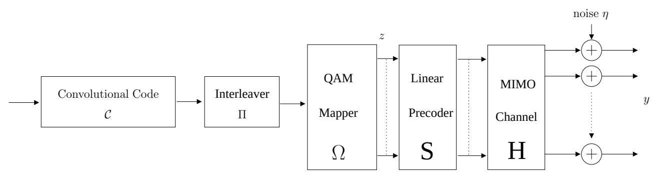

Fig. 1. Space-Time BICM transmitter scheme.

With a vanishing coding rate, i.e. $R_c \to 0$, it is possible to attain the overall system diversity order $n_r n_c n_t$ produced by the receive antennas, the transmit antennas and the distinct channel states. Unfortunately, this is unacceptable due to the vanishing transmitted information rate. Precoding is one means to achieve maximum diversity with a non-vanishing coding rate. Under linear precoding that spreads QAM symbols over s time periods, the Singleton bound becomes [9]:

$$d \leq s n_r \left(\lfloor n_c n_t / s (1 - R_c)\rfloor + 1\right) \quad (5)$$

Now if $R_c = 1$, from the above inequality, we observe that precoding may achieve maximal diversity $n_c n_t$ without the use of error-correcting codes. Unfortunately, near outage performance is impossible in this case due to the weak coding gain of all kinds of precoders. The near-outage performance of ST-BICM is a judicious trade-off between error-control coding and linear QAM precoding. Hence, we propose a simple information theoretical design of multi-dimensional rotations that take into account the interaction between channel coding and symbol space-time spreading.

B. New Design Method: Information Outage Minimization

For a fixed rotation S and n_c fixed MIMO channel matrices H_i, $i = 1 \dots n_c$, defined by the n_c fading blocks, let $\mathcal{I}_{SH} = I(z; y)$ denote the average mutual information of the equivalent channel with QAM input z and complex output y as in (1). The expression of $\mathcal{I}_{SH}$ is

$$\mathcal{I}_{SH} = s.m.n_t - \frac{1}{n_c}\sum_{i=1}^{n_c} E_{z,y|SH_i}\left[\log_2\left(\frac{\sum_{z'} p(y|z', SH_i)}{p(y|z, SH_i)}\right)\right] \quad (6)$$

where $E_{z,y|SH_i}$ is the conditional mathematical expectation over z and y. The channel likelihood is written in its classical form

$$p(y|z, SH) \propto \exp\left(-\frac{\|y - zSH\|^2}{2\sigma^2}\right) \quad (7)$$

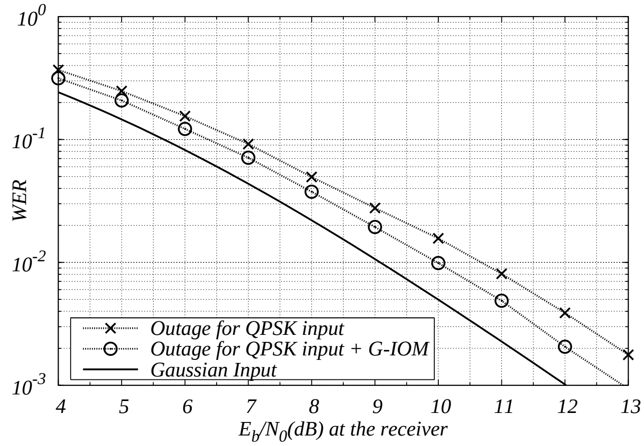

Fig. 2. Outage limits for $n_t = n_c = s = 2$, $n_r = 1$, and $R_c = 1/2$.

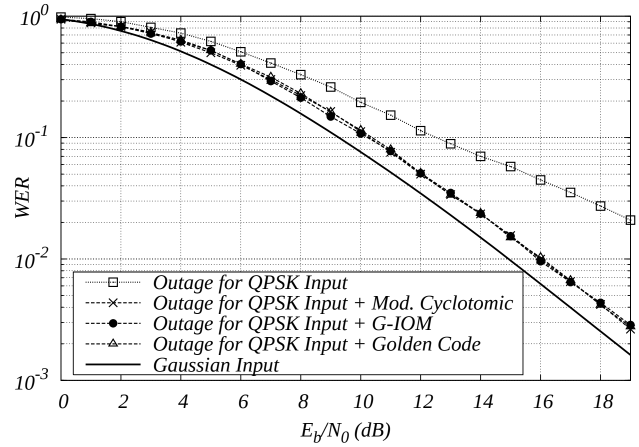

Fig. 3. Outage limits for $n_t = n_c = s = 2$, $n_r = 1$, and $R_c = 3/4$.

Expression (6) assumes that the precoder S does space-time spreading within the same fading block H_i. Its main role is to collect transmit diversity. Time diversity n_c is collected by the convolutional code whereas receive diversity is naturally collected by the detector. The information rate transmitted by the space-time BICM is $R = s.m.n_t.R_c$ bits per s time periods. An outage occurs if the instantaneous capacity, i.e. $\mathcal{I}_{SH}$ in our case, is less than R. The outage probability associated to the rotation S at a given signal-to-noise ratio is

$$P_{out}(S) = P\left(\mathcal{I}_{SH} < s.m.n_t.R_c\right) \quad (8)$$

The new design, called IOM (Information Outage Minimization), selects a matrix S_{IOM} within the ensemble $\aleph$ of random unitary matrices such that

$$S_{IOM} = \arg \min_{S \in \aleph} P_{out}(S) \quad (9)$$

As an example, choosing the best rotation within an ensemble $\aleph$ limited to 2000 matrices yields the matrix written below, for QPSK alphabet with $n_t = s = 2$ and coding rate $R_c = 1/2$

$$S_{IOM} = \begin{bmatrix} 0.57e^{+j1.71} & 0.64e^{+j1.55} & 0.14e^{-j1.89} & 0.49e^{+j1.22} \\ 0.34e^{-j0.94} & 0.51e^{+j2.82} & 0.57e^{+j1.26} & 0.54e^{+j0.27} \\ 0.59e^{-j1.38} & 0.04e^{-j0.04} & 0.61e^{-j1.46} & 0.52e^{+j1.25} \\ 0.46e^{-j0.84} & 0.57e^{+j1.74} & 0.53e^{+j3.05} & 0.43e^{-j2.66} \end{bmatrix}$$

A smaller set $\aleph_G$ of random unitary matrices is obtained by adding to $\aleph$ the first genie constraint, i.e. orthogonal sub-rows in S. This second design, called G-IOM, selects a matrix S_{G-IOM} satisfying

$$S_{G-IOM} = \arg \min_{S \in \aleph_G} P_{out}(S) \quad (10)$$

As an example, choosing the best rotation within an ensemble $\aleph_G$ limited to 2000 matrices yields the matrix written below, for QPSK alphabet with $n_t = s = 2$ and coding rate $R_c = 1/2$

$$S_{G-IOM} = \begin{bmatrix} 0.88e^{-j0.30} & 0 & 0 & 0.48e^{-j0.55} \\ 0.48e^{-j0.33} & 0 & 0 & 0.88e^{+j2.57} \\ 0 & 0.47e^{-j2.12} & 0.88e^{+j2.85} & 0 \\ 0 & 0.88e^{+j2.96} & 0.47e^{-j1.49} & 0 \end{bmatrix}$$

In a similar fashion, a DNA-IOM precoder minimizes the information outage and satisfies DNA constraints [10]. The matrix S_{DNA} given below corresponds to $n_t = 4$ and $s = 2$. The DNA-IOM precoder is obtained by combining S_{DNA} with $\xi_{DNA-IOM}$. Also, the DNA cyclotomic precoder is constructed by combining S_{DNA} to $\xi_{DNA-Cyclo} = S_{Cyclo}$ given previously in (3).

$$S_{DNA} = \begin{bmatrix} \xi_{11} & \xi_{12} & 0 & 0 & \xi_{13} & \xi_{14} & 0 & 0 \\ 0 & 0 & \xi_{11} & \xi_{12} & 0 & 0 & \xi_{13} & \xi_{14} \\ \xi_{21} & \xi_{22} & 0 & 0 & \xi_{23} & \xi_{24} & 0 & 0 \\ 0 & 0 & \xi_{21} & \xi_{22} & 0 & 0 & \xi_{23} & \xi_{24} \\ \xi_{31} & \xi_{32} & 0 & 0 & \xi_{33} & \xi_{34} & 0 & 0 \\ 0 & 0 & \xi_{31} & \xi_{32} & 0 & 0 & \xi_{33} & \xi_{34} \\ \xi_{41} & \xi_{42} & 0 & 0 & \xi_{43} & \xi_{44} & 0 & 0 \\ 0 & 0 & \xi_{41} & \xi_{42} & 0 & 0 & \xi_{43} & \xi_{44} \end{bmatrix}$$

$$\xi_{DNA-IOM} = \begin{bmatrix} 0.73e^{-j0.81} & 0.22e^{+j4.62} & 0.15e^{+j0.60} & 0.61e^{+j2.59} \\ 0.21e^{+j3.99} & 0.56e^{+j4.44} & 0.62e^{+j0.25} & 0.50e^{-j1.29} \\ 0.57e^{+j0.79} & 0.13e^{-j1.28} & 0.57e^{-j0.63} & 0.57e^{+j0.90} \\ 0.29e^{+j1.01} & 0.78e^{+j3.49} & 0.51e^{+j2.27} & 0.20e^{+j0.91} \end{bmatrix}$$

Figures 2 and 3 show the outage limit for different type of precoders in terms of Word Error Rate versus signal-to-noise ratio. The outage probability has been also evaluated for other system parameters. Surprisingly, algebraic precoders satisfying the first genie condition perform as good as G-IOM precoding. Cyclotomic rotation and Golden code are superimposed with G-IOM in Figure 2. All outage evaluations have been made by (6) and (8), without gaussian and analytical approximations when the channel input is a gaussian alphabet as in [18][20].

IV. Simulation results and Conclusions

In order to emphasize the diversity order created by coding at the transmitter side, all computer simulations have been conducted with the number of receive antennas $n_r = 1$. Figures 4 and 5 illustrate the word error rate performance of a space-time BICM for $n_t = 2$ transmit antennas, $n_c = 2$ channel states, $s = 2$ time period spreading and a coding rate $R_c = 1/2$. Figure 4 illustrates the case with $n_t = 4$ transmit antennas and a precoding spread factor $s = 2$. At the first iteration, for $n_t = 2$, IOM precoding slightly outperforms other rotations. After 10 detection/decoding iterations, IOM

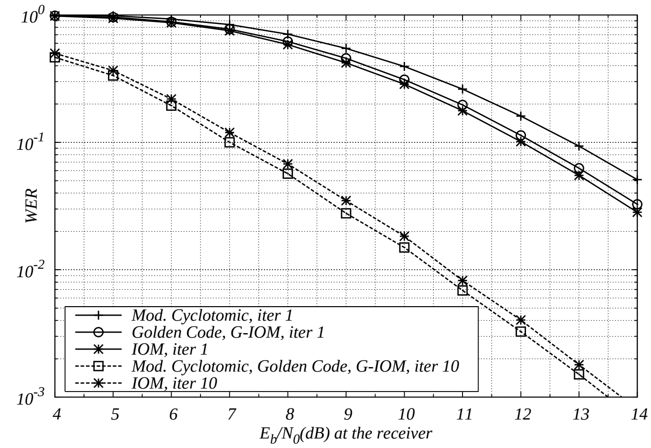

Fig. 4. Word error rate of ST-BICM at iterations 1 and 10. MIMO channel $n_t = s = n_c = 2$, $n_r = 1$. Interleaver size $N = 2048$ bits, rate 1/2 16-state $(23, 35)$ convolutional code. QPSK modulation.

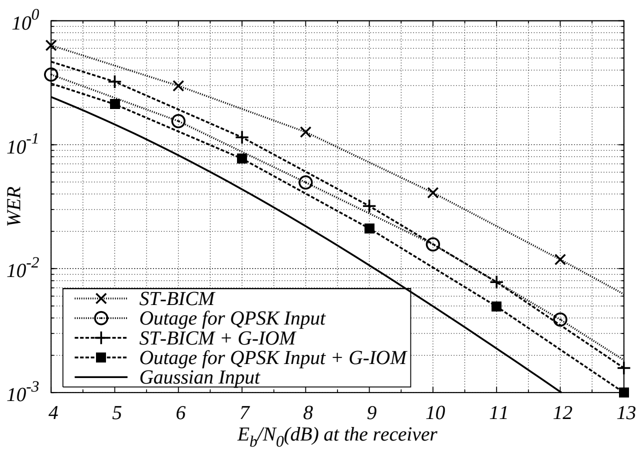

Fig. 5. Word error rate of ST-BICM versus outage limit. MIMO channel $n_t = s = n_c = 2$, $n_r = 1$. Interleaver size $N = 2048$ bits, rate 1/2 16-state $(23, 35)$ convolutional code. QPSK modulation.

is outperformed by G-IOM and other algebraic rotations. The slight difference in performance is still apparent for $n_t = 4$.

From the observed performance and the flexibility of linear precoders, we conclude the following. Cyclotomic rotations satisfying genie/DNA conditions are the best choice for precoding in space-time bit-interleaved coded modulations. These rotations are optimal in both algebraic and information theoretical senses. They exist for any set of MIMO channel parameters, mainly the number of transmit antennas and the precoder time-spreading factor.

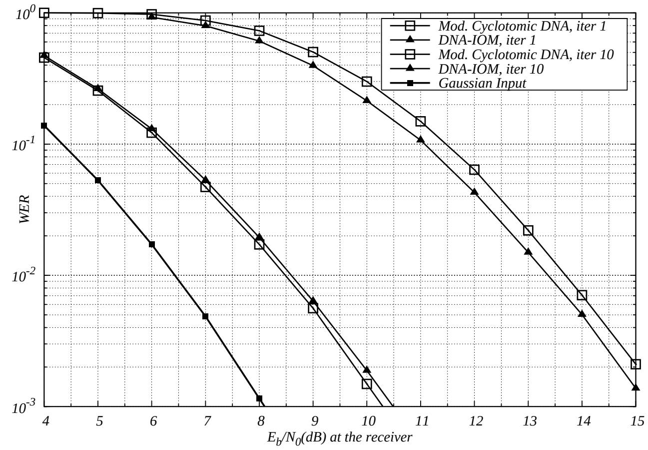

Fig. 6. Word error rate of ST-BICM at iterations 1 and 10 versus outage limit with gaussian input. MIMO channel $n_t = 4$, $s = n_c = 2$, $n_r = 1$. Interleaver size $N = 2048$ bits, rate 1/2 16-state $(23, 35)$ convolutional code. QPSK modulation.

REFERENCES

[1] G. Battail, "Rotating a redundant constellation in signal space against channel fluctuations," in *Proc. Int. Conf. on Communications Technology*, Beijing, China, July 1989 (postponed until 1990).

[2] J.-C. Belfiore, G. Rekaya and E. Viterbo, "The Golden code: A 2x2 full-rate space-time code with nonvanishing determinants," *IEEE Trans. on Inf. Theory*, vol. 51, no. 4, pp. 1432-1436, April 2005.

[3] K. Boullé and J.-C. Belfiore, "Modulation scheme designed for the Rayleigh fading channel", in *Proc. of CISS'92*, Princeton NJ, pp. 288-293, Mar. 1992.

[4] J.J. Boutros and E. Viterbo, "Signal Space Diversity: a power and bandwidth efficient diversity technique for the Rayleigh fading channel," *IEEE Trans. on Inf. Theory*, vol. 44, no. 4, pp. 1453-1467, July 1998.

[5] J. Boutros, N. Gresset, L. Brunel, "Turbo coding and decoding for multiple antenna channels," *Int. Symp. on Turbo Codes*, Brest, Sept. 2003. Downloadable at *http://www.comelec.enst.fr/∼boutros/coding*

[6] G. Caire, G. Taricco and E. Biglieri, "Bit-interleaved coded modulation," *IEEE Trans. on Inf. Theory*, vol. 44, no. 3, pp. 927-946, May 1998.

[7] V. M. DaSilva and E.S. Sousa, "Fading-resistant modulation using several transmitter antennas", *IEEE Trans. on Comm.*, vol. 45, no. 10, pp. 1236-1244, Oct. 1997.

[8] H. El Gamal and M. O. Damen, "Universal space-time coding," *IEEE Trans. on Inf. Theory*, vol. 49, pp. 1097-1119, May 2003.

[9] N. Gresset, J.J. Boutros and L. Brunel, "Optimal linear precoding for BICM over MIMO channels," *IEEE Int. Symp. on Inf. Theory*, pp. 66, June 2004.

[10] N. Gresset, "New space-time coding techniques with bit-interleaved coded modulations," Ph.D. thesis, ENST, Paris, Dec. 2004, downloadable at *http://www.comelec.enst.fr/∼boutros/coding*.

[11] J.-C. Guey, M.P. Fitz, M.R. Bell, and W.-Y. Kuo, "Signal design for transmitter diversity wireless communication systems over Rayleigh fading channels," in *Proc. Vehicular Technology Conf. (VTC'96)*, Atlanta, GA, Apr. 1996.

[12] A. Guillén i Fàbregas, "Concatenated Coded for Block-Fading Channels", Ph.D. thesis, EPFL, Lausanne, July 2004, downloadable at *http://www.comelec.enst.fr/∼boutros/coding*.

[13] B. Hassibi and B.M. Hochwald, "High-rate codes that are linear in space and time," *IEEE Trans. on Inf. Theory*, vol. 48, no. 7, pp. 1804-1824, July 2002.

[14] R. Knopp and P. Humblet, "On coding for block fading channels," *IEEE Trans. on Inf. Theory*, vol. 46, no. 1, pp. 189-205, Jan. 2000.

[15] C. Lamy and J. Boutros, "On Random Rotations Diversity and Minimum MSE Decoding of Lattices," *IEEE Trans. on Information Theory*, vol. 46, pp. 1584-1589, July 2000.

[16] D. Rainish, "Diversity Transform for Fading Channels," *IEEE Trans. on Inf. Theory*, vol. 44, no. 12, pp. 1653-1661, Dec. 1996.

[17] B.A. Sethuraman, B.S. Rajan, and V. Shashidhar, "Full-diversity, high-rate space-time block codes from division algebras," *IEEE Trans. on Inf. Theory*, vol. 49, no. 10, pp. 2596-2616, Oct. 2003.

[18] P.J. Smith and M. Shafi , "On a Gaussian approximation to the capacity of wireless MIMO systems," in *Proc. of the IEEE Int. Conference on Communications*, vol. 1, pp. 406-410, May 2002.

[19] V. Tarokh, N. Seshadri, and A.R. Calderbank, "Space-Time Codes for High Data Rate Wireless Communication: Performance Criterion and Code Construction," *IEEE Trans. on Inf. Theory*, Vol. 44, No. 2, Mar. 1998.

[20] Z. Wang and G.B. Giannakis, "Outage mutual information of space-time MIMO channels," *IEEE Trans. on Inf. Theory*, vol. 50, no. 4, pp. 657-662, April 2004.