FINAL EXAM
ECEN 478 - Wireless Communications

Joseph J. Boutros

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Part I. Receivers for Code Division Multiple Access

We consider a synchronous DSSS-CDMA system with 2 users. The BPSK symbol of user 1 (resp. user 2) is denoted by $I_1 = \pm\omega_1$ (resp. $I_2 = \pm\omega_2$). Each symbol is spread by a factor of N after multiplication with N chips of a pseudo-noise (PN) sequence. The two users are using two different PN sequences with a cross-correlation equal to ρ, where $0 \leq \rho < 1$. The real wireless channel model is

$$y_i = x_i^1 + x_i^2 + \eta_i, \tag{1}$$

where the time index i varies from 0 to $N-1$, y_i is the sample received at the CDMA detector input, x_i^j is the chip transmitted by user j at time i, $j = 1, 2$. It is given by

$$x_i^j = s_i^j \cdot I_j, \tag{2}$$

where $s^j = (s_0^j \; s_1^j \; \ldots \; s_{N-1}^j)$ is the spreading sequence (signature) of user j, $s_i^j = \pm 1$. As stated above, the two signatures satisfy

$$\frac{1}{N}\sum_{i=0}^{N-1} s_i^1 \cdot s_i^2 \;=\; \rho. \tag{3}$$

The real additive white noise η_i in (1) is $\mathcal{N}(0, \sigma^2)$, by convention we assign $\sigma^2 = N_0/2$. The signal-to-noise ratios are defined as (per user per information bit)

$$\gamma_j = N\frac{(\omega_j)^2}{N_0}, \qquad j = 1, 2. \tag{4}$$

The CDMA joint receiver starts by despreading both users. The despreader output is

$$r_1 \;=\; \frac{< s^1, y >}{N} \;=\; \frac{1}{N}\sum_{i=0}^{N-1} s_i^1 \cdot y_i, \tag{5}$$

when despreading the first user. Similarly, we have

$$r_2 \;=\; \frac{< s^2, y >}{N} \;=\; \frac{1}{N}\sum_{i=0}^{N-1} s_i^2 \cdot y_i, \tag{6}$$

when despreading the second user. The symbol "$<, >$" denotes the scalar product of two vectors, and the detector observation vector before despreading is $y = (y_0 \; y_1 \; \ldots \; y_{N-1})$.

In column notations, the symbol vector, the observation vector after despreading, and the noise vector after despreading are

$$I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, \quad r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Finally, we define the correlation matrix of the two signatures as

$$R \;=\; \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

I.1) Briefly sketch the proof of the following expression

$$r = RI + b. \tag{7}$$

Relate b to η.

I.2) Express the variance $\sigma_b^2 = E[|b_1|^2] = E[|b_2|^2]$ function of the original channel noise variance σ^2 and the spreading factor N.

I.3) Compute the correlation coefficient $E[b_1 b_2]$. Under what condition the two noise samples b_1 and b_2 are uncorraleted?

Let Δ be the energy ratio of the two users,

$$\Delta = \frac{\gamma_2}{\gamma_1} = \left(\frac{\omega_2}{\omega_1}\right)^2$$

It is supposed that $\rho^2 \leq \Delta \leq 1/\rho^2$. Let us also define the following function

$$F(\gamma,\Delta) = \frac{1}{2}\mathbb{Q}\left(\sqrt{2\gamma(1+\rho\sqrt{\Delta})^2}\right) + \frac{1}{2}\mathbb{Q}\left(\sqrt{2\gamma(1-\rho\sqrt{\Delta})^2}\right), \tag{8}$$

where $\mathbb{Q}(x)$ is the gaussian tail function.

Conventional detection

A conventional receiver does not take into account the multiple access interference. In order to estimate I_1, a conventional detector simply checks the sign of r_1,

$$\hat{I}_1(conv) = \Psi(r_1) = \begin{cases} +\omega_1, & r_1 > 0, \\ -\omega_1, & r_1 < 0. \end{cases}$$

Similarly, I_2 is estimated from r_2. The correlation ρ between s^1 and s^2 is not considered by such a conventional detection (in opposition to multiuser detection, as the two other techniques described below).

I.4) Under conventional detection, prove that the error probability of the first user is

$$P_{e1}(conv) = F(\gamma_1, \Delta) \tag{9}$$

where $F(,)$ is defined by (8). In an identical manner, prove that the second user has

$$P_{e2}(conv) = F(\gamma_2, \frac{1}{\Delta}) \tag{10}$$

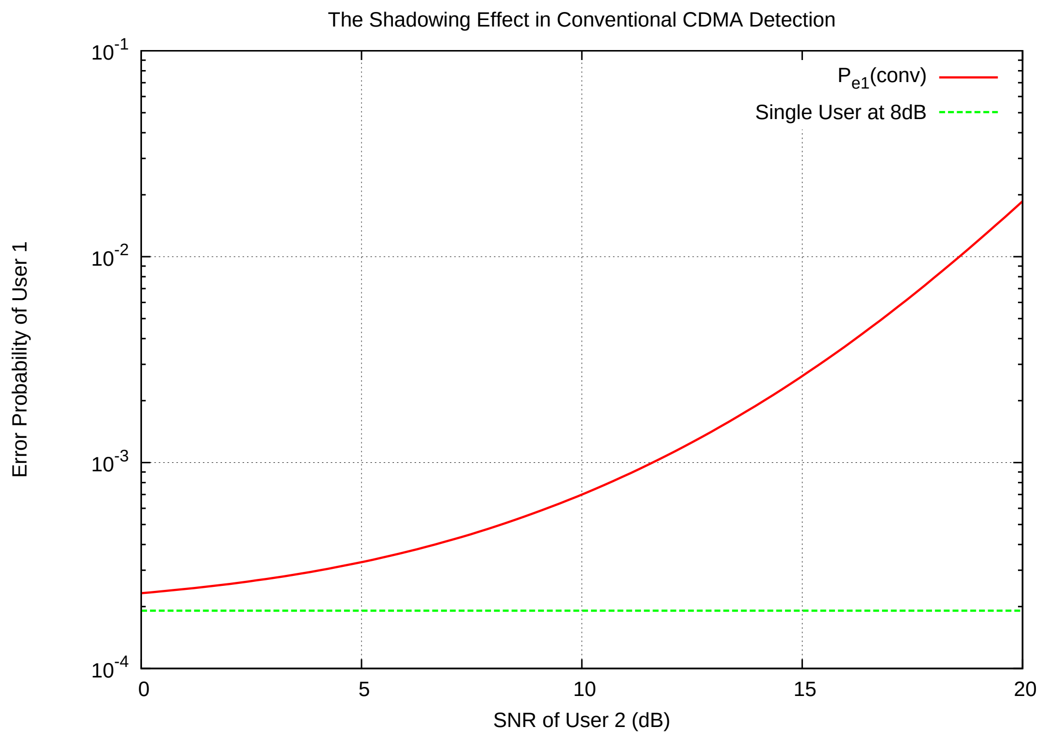

Figure 1: Conventional CDMA detection of two users. Error probability of User 1 versus the SNR of User 2. Parameters are $\gamma_1 = 8dB$ and $\rho = 1/8$.

I.5) Assume that γ_1 and ρ are fixed. Let the second user increase γ_2, how does $P_{e1}(conv)$ behave? Interpret the result illustrated in Figure 1 for $\gamma_1 = 8dB$ and $\rho = 1/8$.

Zero-forcing detection

The multiuser ZF receiver multiplies the observation given in (7) by the inverse of the correlation matrix,

$$\tilde{r} = R^{-1}r = I + R^{-1}b$$

Then, threshold detection is applied on $\tilde{r}$,

$$\hat{I}_1(ZF) = \Psi(\tilde{r}_1) = \begin{cases} +\omega_1, & \tilde{r}_1 > 0, \\ -\omega_1, & \tilde{r}_1 < 0. \end{cases}$$

I.6) Find the variance per component $\tilde{\sigma}^2$ of $\tilde{b} = R^{-1}b$ as a function of ρ and σ_b^2.

I.7) Show that $\tilde{\sigma}^2 \geq \sigma_b^2$. Thus, the ZF receiver eliminates the multiple access interference but it amplifies the additive noise. Interpret the result illustrated in Figures 2 and 3. Is there any shadowing effect after ZF detection?

I.8) Under ZF detection, prove that User 1 error probability is

$$P_{e1}(ZF) = \mathbb{Q}\left(\sqrt{2\gamma_1(1-\rho^2)}\right) \tag{11}$$

The value of the above expression has been plotted in Figure 2 and Figure 3 for $\rho = 1/8$ and $\rho = 1/2$ respectively.

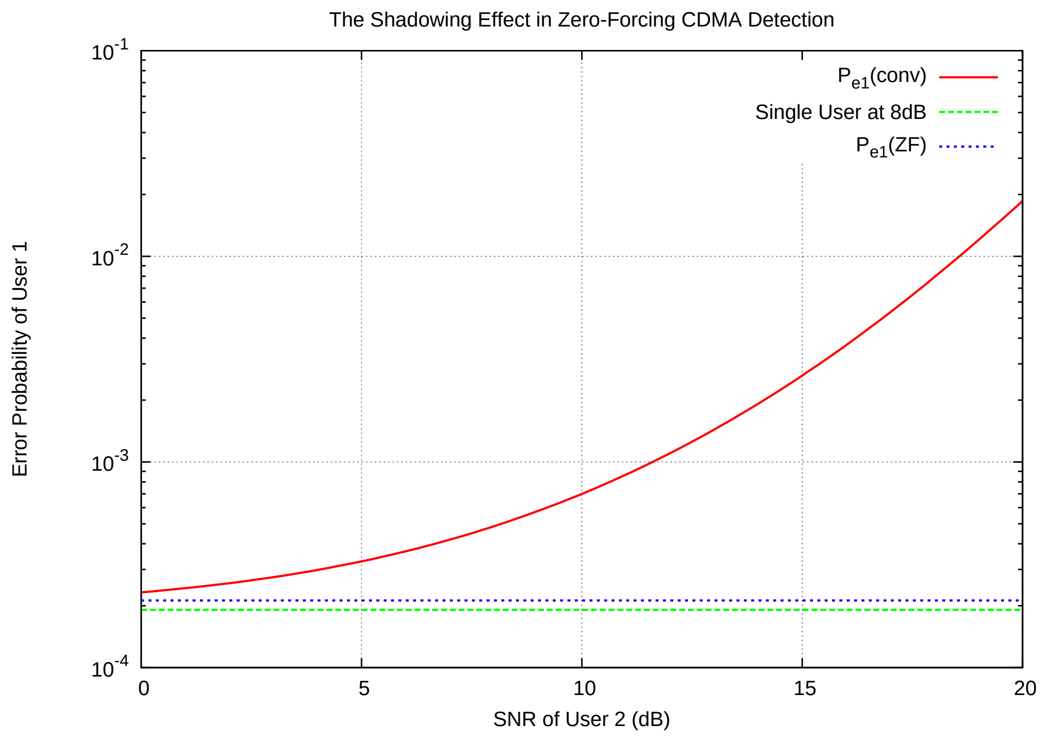

Figure 2: ZF versus conventional CDMA detection. Error probability of User 1 versus the SNR of User 2. Parameters are $\gamma_1 = 8dB$ and $\rho = 1/8$. Noise amplification due to ZF is negligible at this value of the cross-correlation ρ.

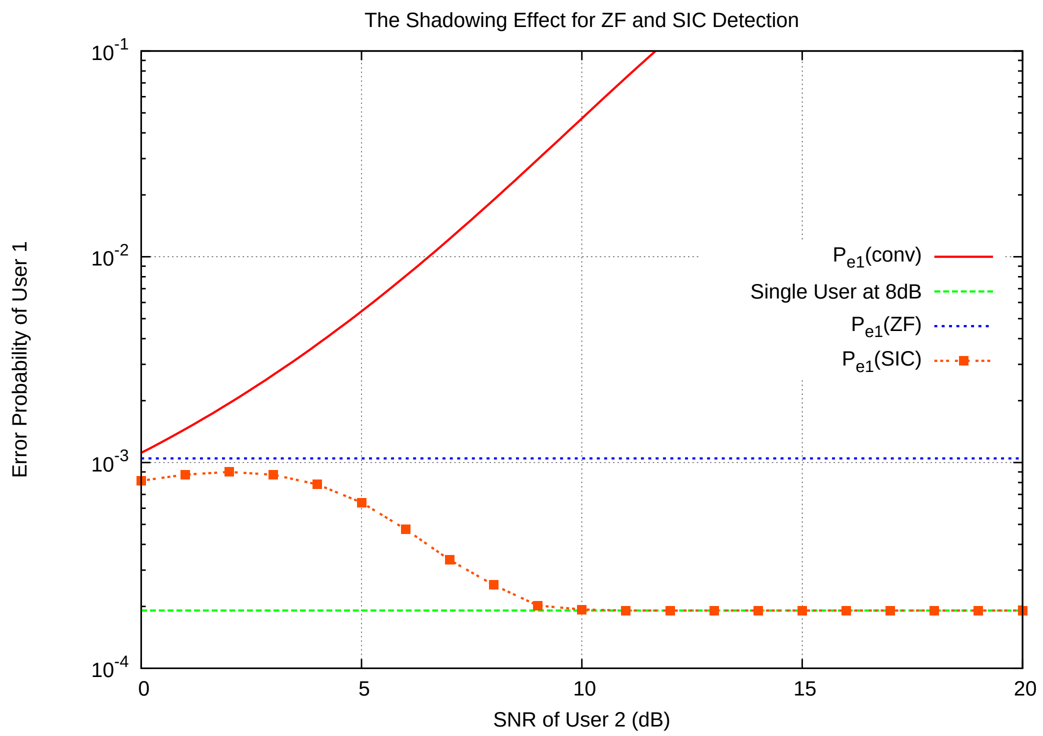

Figure 3: SIC and ZF versus conventional CDMA detection. Error probability of User 1 versus the SNR of User 2. Parameters are $\gamma_1 = 8dB$ and $\rho = 1/2$. Noise amplification due to ZF is not negligible for $\rho = 1/2$.

Subtractive interference cancellation
The SIC detector proceeds as follows, in an iterative manner,
Iteration 1:

- Consider $\tilde{r}_1 = r_1 = I_1 + \rho I_2 + b_1$.
- Find $\hat{I}_1 = \Psi(\tilde{r}_1)$.
- Compute $\tilde{r}_2 = r_2 - \rho \hat{I}_1 = \rho(I_1 - \hat{I}_1) + b_2$.
- Find $\hat{I}_2 = \Psi(\tilde{r}_2)$.

Iteration 2:

- Compute $\tilde{r}_1 = r_1 - \rho \hat{I}_2$.
- Find $\hat{I}_1 = \Psi(\tilde{r}_1)$.
- Compute $\tilde{r}_2 = r_2 - \rho \hat{I}_1$.
- Find $\hat{I}_2 = \Psi(\tilde{r}_2)$.

Iterations $m \geq 3$ are identical to iteration 2.
I.9) Let $P_{e1}^{m}(SIC)$ and $P_{e2}^{m}(SIC)$ denote the error probability of User 1 and User 2 respectively, after m iterations. For $m \geq 2$, show that

$$P_{e1}^{m} = (1 - P_{e2}^{m-1})Q_1 + P_{e2}^{m-1} F_1 \tag{12}$$

where $Q_1 = \mathbb{Q}(\sqrt{2\gamma_1})$ is the single user performance of User 1 (absence of interference) and $F_1 = F(\gamma_1, 4\Delta)$ is the conventional CDMA performance of User 1 when the SNR of the interferer is $4\gamma_2$. Show also that

$$P_{e2}^{m} = (1 - P_{e1}^{m})Q_2 + P_{e1}^{m} F_2 \tag{13}$$

where $Q_2 = \mathbb{Q}(\sqrt{2\gamma_2})$ and $F_2 = F(\gamma_2, 4/\Delta)$.
I.10) In the steady state, when $m \to +\infty$, prove that the SIC detector yields

$$P_{e1}(SIC) = \frac{Q_1 + (F_1 - Q_1)Q_2}{1 - (F_1 - Q_1)(F_2 - Q_2)} \tag{14}$$

Under what condition we have $P_{e1}(SIC) \approx Q_1$. In this case, the SIC is capable of eliminating the multiple access interference without noise amplification and without shadowing effect when $\gamma_2 > \gamma_1$. An illustration of SIC behavior is given in Figure 3.

Part II. Multipath Diversity in Spread Spectrum

The notations of Part I are adopted. Only one user is transmitting on a wireless channel using a BPSK modulation and direct sequence spreading via a signature s^1. Time index (at the chip level) is always indicated by the integer i. In this section, the index j refers to time shifts due to multipath. For simplicity of notations, superscripts indicating the user number are eliminated. Hence, the transmitted chips x_i^1 are replaced by x_i, the signature chips s_i^1 are replaced by s_i, and the spreading sequence s^1 is simply $s = \{s_i\}$, $s_i = \pm 1$, and $i \in \mathbb{Z}$. It is assumed that s is periodic with period $N \gg 1$. The auto-correlation of s is supposed to be perfect, i.e.,

$$\rho_s(j) = \frac{1}{N} \sum_{i=0}^{N-1} s_i \cdot s_{i-j} \;\approx\; 0, \quad \forall j \neq 0. \tag{15}$$

Before spreading, the user symbol at time t (at the information bit level) is $I_1[t] = \pm\omega$. The signal-to-noise ratio $\gamma = \gamma_1$ is still defined by (4) in Part 1. In this section, we are only interested by the transmission and the detection of symbol $I_1 = I_1[0]$. The influence of $I_1[-1]$ (past symbol) and $I_1[1]$ (future symbol) are neglected.

The channel is assumed to have multipath fading. Its complex model is

$$y_i = \sum_{j=0}^{L-1} h_j \cdot x_{i-j} + \eta_i, \quad i = 0 \ldots N-1, \tag{16}$$

where $h_j \sim \mathcal{CN}(0,1)$, and $\eta_i \sim \mathcal{CN}(0, 2\sigma^2)$. The L fading coefficients are supposed to be uncorrelated. The transmitted chip is obtained as in (2) via $x_i = s_i I_1$. The number L of channel paths satisfies $1 \leq L \ll N$.

II.1) Assume that the channel coherence bandwidth is $B_{coh} = 500KHz$. Without a spreaded spectrum ($N = 1$), what would be the value of L if the transmitted signal bandwidth is $W_0 = 100KHz$?

II.2) After spreading, e.g., $N = 100$ and $W = NW_0 = 10MHz$, what is the number L of paths observed by the receiver? From a diversity point of view, is it better to have $L = 1$ or $L > 1$?

We would like to build a receiver capable of exploiting the L degrees of freedom in the channel. In order to obtain a diversity of order L, the receiver should create the χ^2 distributed quantity $\sum_{\ell=0}^{L-1} |h_\ell|^2$. This is possible with the structure proposed below, a structure known as the Rake receiver. We propose to build a Rake with L fingers, the output of finger ℓ is denoted by f_ℓ.

II.3) The finger output number ℓ of a Rake receiver is determined by the projection of the received signal on an ℓ-shifted version of the spreading sequence, $\ell = 0, \ldots, L-1$. The finger output is

$$f_\ell = \frac{< \{s_{i-\ell}\}, y >}{N} = \frac{1}{N} \sum_{i=0}^{N-1} s_{i-\ell} y_i, \tag{17}$$

where y_i is given by the channel model in (16). Show that

$$f_\ell \approx h_\ell I_1 + b_\ell, \tag{18}$$

where b_ℓ is an additive gaussian noise.

II.4) After computing the output of its L fingers, the Rake performs a maximum ratio combining

$$U = \sum_{\ell=0}^{L-1} h_\ell^* f_\ell \tag{19}$$

The information symbol is estimated with $\hat{I}_1 = \Psi(\Re\{U\})$. At high SNR, show that the error probability is written as $P_e \approx \tau/\gamma^L$. Find the expression of τ.

II.5) The Rake receiver described above knows perfectly the values of the channel coefficients $\{h_j\}$, i.e., this is a coherent Rake receiver. Is it possible to build a non-coherent Rake receiver?

. *.Good Luck.*

. *.Joseph Boutros.*
