# Performance of optimal codes on Gaussian and Rayleigh fading channels : a geometrical approach. *

Sandrine Vialle
Centre de Recherche Motorola
Espace Technologique
Commune de St Aubin
91193 Gif sur Yvette, France
vialle@com.enst.fr

Joseph Boutros
Communications Department
ENST University
46 Rue Barrault
75013 Paris, France
boutros@com.enst.fr

revised, October 1999

**Abstract**

The error probability of optimal codes with a given length $n$ and a given rate $R$ is estimated on additive white Gaussian noise channels with and without flat fading. A lower bound is given for the error probability on the Gaussian channel and a good approximation is derived in the presence of Rayleigh fading. The performance analysis is restricted to binary codes associated to a binary phase shift keying. However, the results can be easily generalized to any coded modulation.

## 1 Introduction

The channel coding theorem (e.g. see [8]) states that Gallager's exponent $E(R)$ is positive iff the code rate is smaller than capacity. Since the word error probability is upper bounded by $e^{-nE(R)}$, the latter can be used as an average performance estimation for the ensemble of random codes when $n$ is finite and $R$ has a fixed valid value. Instead of using the random coding technique, Shannon gave a lower bound [7] on the performance of all spherical codes in Gaussian channel based on a geometrical approach. Some modifications and improvements of his work have been done recently [1][5]. Nevertheless, the expression derived by Shannon for the main function $Q(\theta)$ is still the simplest one. This function gives the probability of moving outside a cone with half-angle $\theta$. Its asymptotic expression is very accurate: for a code length greater than 100, it is too close to the exact optimal performance. This has been validated numerically by comparing $Q(\theta)$ to the upper bound derived by Shannon in the same paper [7].

In our paper, we generalize Shannon's geometrical approach to the Rayleigh fading channel. Section 3 summarizes the main results on the Gaussian channel and gives some important asymptotic expressions for the solid angle and the code rate. In section 4, we explain how to compute the conditional solid angle in the presence of independent flat fading. Finally, some numerical results are illustrated in the last section. The drawn curves can be used to check the performance of concatenated block and convolutional codes or any other type of codes capable of approaching channel capacity.

*Presented at the 37th Annual Allerton Conference, Illinois.

## 2 System model and notations

A linear binary code $[n, k]_2$ of length $n$ and dimension $k$ is considered. A codeword $c = (c_1, c_2, \ldots, c_n)$ is transmitted over an additive white gaussian noise (AWGN) channel using a binary phase shift modulation (BPSK). The symbols take the values $c_i = \pm\sqrt{2E_c}$, where $E_c$ is the average energy per coded bit. The code rate is $R = k/n$ bits per dimension and the average energy per information bit is given by $E_b = E_c/R$.
Two channel models are studied: the Gaussian channel $y_i = c_i + z_i$ where $z_i$ is a real Gaussian noise with zero mean and variance $N_0$, and the flat fading channel [6] $y_i = \alpha_i c_i + z_i$ where $\alpha_i \in [0, +\infty[$ is Rayleigh distributed with probability density function $p(\alpha_i) = 2\alpha_i e^{-\alpha_i^2}$. Note that $E[\alpha_i^2] = 1$ so the average energy value is not changed by the Rayleigh channel.
Let us now make a geometrical representation of the code set in the $n$-dimensional real space $\mathbb{R}^n$. We assume that the codewords are equally probable and selected at random from the ensemble $\{\pm\sqrt{2E_c}\}^n$. Thus, the codewords $c$ are uniformly distributed on the $n$-dimensional sphere $Sph(O, n)$ centered at the origin with squared radius $E = 2nE_c$. The total space solid angle is denoted $\Omega_n$. The solid angle of a circular cone with half angle $\theta$ is denoted $\Omega(\theta)$. The codeword $c$ is represented as a point lying on $Sph(O, n)$ and corresponding to a solid angle $\Omega(\theta_0) = \Omega_n/M$, where $M = 2^{nR}$ is the code size and $R = R(\theta_0)$.
Figure 1 shows the codeword $c$ surrounded by a circular cone $C(c)$ delimited by an $n-1$-dimensional sphere (circle) $Sph(a, n-1)$ and a pyramid $V(c)$ drawn in an hexagonal form. $V(c)$ is the Voronoi region associated to the codeword $c$, i.e. $V(c)$ is the decision region of $c$ when maximum likelihood (ML) decoding is applied. The circle $Sph(a, n-1)$ is the intersection of the sphere $Sph(O, n)$ and a hyperplane denoted by $\mathcal{H}$. The radius $\rho_0 = \rho_0(\theta_0)$ of the circle is chosen such that the solid angle $\Omega(\theta_0)$ of the pyramid $V(c)$ is equal to that of the cone $C(c)$ delimited by $Sph(a, n-1)$. Notice that the projection of the codeword point $c$ on the hyperplane $\mathcal{H}$ is denoted $a$. Two other radii are also shown on figure 1, the packing radius $\rho_p$ and the covering radius $\rho_c$ of the spherical code equivalent to the considered linear binary code. The vector modules of $c$ and $a$ are related by $\|a\| = \|c\| \cos(\theta_0)$ where $\|c\|^2 = E$.

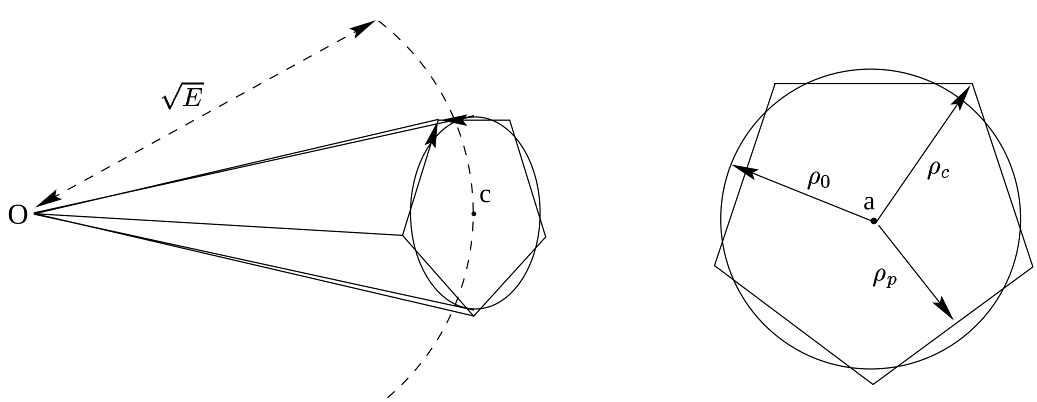


Figure 1: Geometrical representation of the solid angle associated to a codeword.

# 3 Optimal performance on the AWGN channel

The codewords are supposed to be uniformly distributed and have the same a priori probability. By symmetry, we can state that the average word error probability $P_{ew}$ is equal to the conditional error probability $P_e(c)$ when $c$ is transmitted, e.g. $c = (+\sqrt{2E_c}, +\sqrt{2E_c}, \dots, +\sqrt{2E_c})$. The error probability $P_e(c)$ is thus the probability for the received signal to be moved by the Gaussian noise $z$ outside the pyramid $V(c)$.
For all n, the Gaussian noise probability density is a decreasing function of the Euclidean distance $d$:

$$\frac{\partial p_z}{\partial V} = \frac{1}{(2\pi N_0)^{n/2}} \exp(-\frac{d^2}{2N_0}) \tag{1}$$

It can be easily shown that

$$\int_{C(c)} \frac{\partial p_z}{\partial V} dV \geq \int_{V(c)} \frac{\partial p_z}{\partial V} dV \tag{2}$$

Let us denote $Q(\theta)$ the probability for a codeword on the axis of a cone of solid angle $\Omega(\theta)$ being moved outside this cone. We have

$$Q(\theta_0) = 1 - \int_{C(c)} \frac{\partial p_z}{\partial V} dV$$

Using inequality 2 and the above definition of $Q(\theta_0)$, we obtain the following lower bound for the word error probability [7]

$$Q(\theta_0) \leq P_{ew}$$

In the following, we explain how to compute $Q(\theta_0)$ for a given length $n$ and a given rate $R$. The relation $R = R(\theta_0)$ and the expression of $Q(\theta)$ have been developed by C.E. Shannon [7] and may also be applied to any spherical code [3], i.e. any coded modulation with equal energy signals.
Let us denote $\Gamma(x)$ the Euler function defined by $\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt$. The volume of an $n$-dimensional sphere with unit radius is [3]

$$Vol(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}$$

For large space dimensions, $n >> 1$, the Stirling approximation yields

$$Vol(n) \approx \frac{\left(\frac{2e\pi}{n}\right)^{n/2}}{\sqrt{n\pi}}$$

The total solid angle $\Omega_n$ is equal to the surface of the $n$-dimensional sphere considered above and is related to its volume by $\Omega_n = nVol(n)$. The solid angle $\Omega(\theta)$ is computed by summing elementary ring surfaces $d\Omega(\phi) = \Omega_{n-1} \sin^{n-2}\phi d\phi$.

$$\Omega(\theta) = \frac{(n-1)\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \int_0^{\theta} \sin^{n-2}\phi d\phi$$

Now the code rate is related to the half angle by $M = 2^{nR} = \frac{\Omega_n}{\Omega_0}$.

$$R = \frac{1}{n} \log_2 \left( \frac{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \int_0^{\theta_0} \sin^{n-2}\phi d\phi} \right)$$

Asymptotically, when $n >> 1$, the expressions of the solid angle and the rate become

$$\Omega(\theta) \approx \frac{1}{\pi\sqrt{2}}\left(\frac{2e\pi}{n}\right)^{n/2}\frac{\sin^{n-1}\theta}{\cos\theta} \tag{3}$$

$$2^{nR} \approx \frac{\sqrt{2\pi n}\sin\theta_0\cos\theta_0}{\sin^n\theta_0} \tag{4}$$

Finally let us compute $Q(\theta)$ from its derivative

$$Q(\theta) = \int_\theta^{\frac{\pi}{2}} -dQ(\phi) + Q\Big(\frac{\pi}{2}\Big) \tag{5}$$

Notice that $Q\big(\frac{\pi}{2}\big) = \frac{1}{2}erfc\big(\frac{E}{2N_0}\big)$, where $erfc$ is the complementary error function [6]. This term will be neglected since it has a minor influence on $Q(\theta)$.
Consider equation 1 and figure 2. $dV$ is the volume of a ring given by its depth $dr$, its width $rd\theta$ and placed at a distance $d$ from $c$ given by $d^2 = E + r^2 - 2r\sqrt{E}\cos\theta$. The cone element of volume $dV$ is

$$dV = rdrd\theta\frac{(n-1)\pi^{\frac{n-1}{2}}(r\sin\theta)^{n-2}}{\Gamma(\frac{n+1}{2})}$$

The probability density for moving $c$ inside this cone element according to the noise Gaussian distribution is

$$-d^2Q(r,\theta) = \frac{1}{(2\pi N_0)^{n/2}}\exp\left(-\frac{E+r^2-2r\sqrt{E}\cos\theta}{2N_0}\right)\frac{(n-1)\pi^{\frac{n-1}{2}}(r\sin\theta)^{n-2}}{\Gamma\big(\frac{n+1}{2}\big)}rdrd\theta$$

The differential $-dQ(\theta)$ is obtained by integrating $-d^2Q(r,\theta)$ on the variable r, between 0 and $+\infty$:

$$-\frac{dQ(\theta)}{d\theta} = \frac{(n-1)\exp\big(-\frac{E\sin^2\theta}{2N_0}\big)\sin^{n-2}\theta d\theta}{(2N_0)^{n/2}\sqrt{\pi}\Gamma\big(\frac{n+1}{2}\big)}\int_0^{+\infty}\exp\left(-\frac{(r-\sqrt{E}\cos\theta)^2}{2N_0}\right)r^{n-1}dr$$

This equation can be approximated asymptotically [7]

$$-dQ(\theta) \approx \frac{(n-1)}{\sqrt{n\pi}\sqrt{1+G^2\sin^2\theta}}\Big[G\sin\theta\exp\Big(-\frac{E_c}{N_0}+\frac{1}{2}\sqrt{\frac{2E_c}{N_0}}G\cos\theta\Big)\Big]^n d\theta \tag{6}$$

where $G = \frac{1}{2}\Big[\sqrt{\frac{2E_c}{N_0}}\cos\theta + \sqrt{\frac{2E_c}{N_0}\cos^2\theta+4}\Big]$. To derive the numerical results versus the signal-to-noise ratio $E_b/N_0$, we used equations (5) and (6) combined together with the relation $E_c = RE_b$ to evaluate the lower bound on the word error probability

$$Q(\theta_0) \approx \frac{1}{\sqrt{n\pi}}\frac{1}{\sqrt{1+G^2}\sin\theta_0}\frac{\Big[G\sin\theta_0\exp\Big(-\frac{E_c}{N_0}+\frac{1}{2}\sqrt{\frac{2E_c}{N_0}}G\cos\theta_0\Big)\Big]^n}{\sqrt{\frac{2E_c}{N_0}}G\sin^2\theta_0-\cos\theta_0} \tag{7}$$

The cone half-angle $\theta_0$ is computed by solving equation (4) for a fixed space dimension $n$ and a fixed rate $R$.

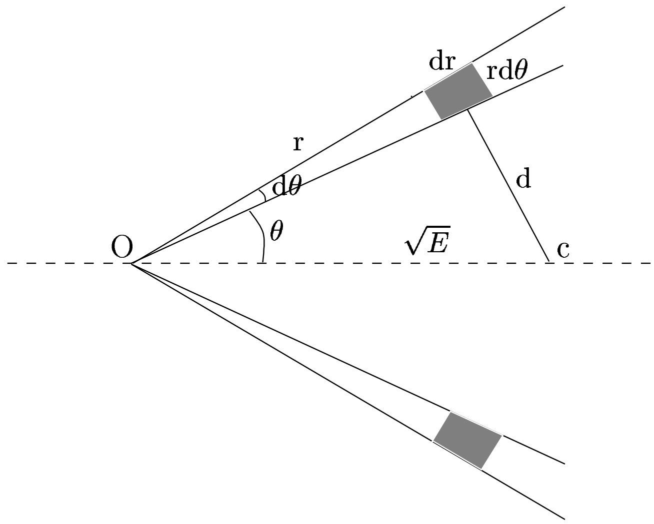


Figure 2: Section of the cone of half-angle $\theta$.

# 4 Optimal performance on the Rayleigh channel

The spherical code is affected by a real fading $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. The codeword point $c$ is transformed into $\alpha c = (\alpha_1 c_1, \alpha_2 c_2, \dots, \alpha_n c_n)$ and then corrupted by the white Gaussian noise. The symmetric Voronoi region $V(c)$ of the spherical code without fading is now transformed into $\alpha V(c)$ which is not completely superimposed with the exact Voronoi region $V(\alpha c)$ of the faded spherical code. The problem of finding the shape and boundaries of $V(\alpha c)$ is intractable especially in high dimensions. Hence, we are only able to find theoretically an approximation (not an exact lower bound) for the word error probability of the linear binary code $[n,k]_2$ over the Rayleigh channel. To do so, we first approximate the solid angle of $V(\alpha c)$ by that associated to $\alpha C(c)$. Then we compute the equivalent angle $\theta_1 = \theta_1(\alpha, \theta_0)$ corresponding to a circular cone having the same solid angle and finally we evaluate the conditional error probability $P_{ew}(\theta_1) \approx Q(\theta_1)$.

According to section 2, the elliptic cone $\alpha C(c)$ is now delimited by the $n-1$-dimensional ellipsoid $\mathcal{E} = \alpha Sph(a, n-1)$. The latter is defined as the intersection of the new hyperplane $\alpha\mathcal{H}$ and the $n$-dimensional ellipsoid centered at the origin $\alpha Sph(O,n)$. All points $x$ belonging to $\mathcal{E}$ satisfy the two following equations

$$\alpha Sph(O,n): \qquad \sum_{i=1}^{n} \frac{x_i^2}{\alpha_i^2} = E \tag{8}$$

and

$$\alpha\mathcal{H}: \qquad \sum_{i=1}^{n} \frac{a_i x_i}{\alpha_i} = \|a\|^2 \tag{9}$$

It is easy to show that the point $\alpha a$ is the center of the ellipsoid $\mathcal{E}$ and that $\alpha a$ belongs to the line $O - \alpha c$.
The solid angle $\Omega(\theta_1)$ of $\alpha C(c)$ is equal to the surface of $\mathcal{E}$ (i.e. $n-1$-dimensional volume)

projected on the $n$-dimensional unity sphere. The surface $\Omega(\theta_1)$ is well approximated by

$$\Omega(\theta_1) \approx Vol(n-1) \prod_{j=1}^{n-1} d_j = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \prod_{j=1}^{n-1} d_j \tag{10}$$

where $\{d_j\}$ are the projections of the $n-1$ axes of $\mathcal{E}$ on the unity sphere. Notice that each ellipsoid axis is defined by two opposite points around the center $\alpha a$. The distance $2d_j$ is the arc length obtained after the projection of the two opposite points. Equation (10) is an approximation and not an equality because the hyperplane $\alpha\mathcal{H}$ is not orthogonal to the line $O - \alpha c$ when $\alpha \neq 1$.

To find the $2n-2$ points defining the $n-1$ axis of $\mathcal{E}$, we optimize $\|x - \alpha\ a\|^2$ under the two constraints (8) & (9). Introducing the two Lagrange multipliers $\lambda$ and $\gamma$ respectively associated to the constraints (8) & (9), the following function $f(x)$ is optimized with respect to $x$

$$f(x) = \|x - \alpha\ a\|^2 - \lambda\Big(\sum_{i=1}^{n} \frac{x_i^2}{\alpha_i^2} - E\Big) - \gamma\Big(\sum_{i=1}^{n} \frac{a_i x_i}{\alpha_i} - \|a\|^2\Big) \tag{11}$$

The solutions to the above optimization (algebraic manipulations and details omitted) are the $2n-2$ points $x = x(\lambda, \gamma) = (x_1, x_2, \ldots, x_n)$ given by

$$x_i = \frac{\frac{\gamma}{2} + \alpha_i^2}{\alpha_i^2 - \lambda} \times \alpha_i a_i \qquad i = 1, \ldots, n \tag{12}$$

where $a_i = c_i cos(\theta_0) = \sqrt{2E_c} cos(\theta_0)$. The multiplier $\lambda$ takes $n-1$ values $\lambda_j$, $j = 1 \ldots n-1$, which are the solutions of the equation

$$\sum_{i=1}^{n} \frac{1}{\alpha_i^2 - \lambda} = 0$$

For each value of $\lambda$, there are two values for $\gamma$ associated to the two opposite points around $\alpha a$ and given by the following second degree equation

$$V_1\gamma^2 + 4V_2\gamma + 4\left(V_4 - \frac{1}{\cos^2\theta_0}\right) = 0 \tag{13}$$

where

$$\begin{aligned} V_1 &= \sum_{i=1}^{n} \frac{1}{(\alpha_i^2 - \lambda)^2} \\ V_2 &= \sum_{i=1}^{n} \frac{\alpha_i^2}{(\alpha_i^2 - \lambda)^2} \\ V_4 &= \sum_{i=1}^{n} \frac{\alpha_i^4}{(\alpha_i^2 - \lambda)^2} \end{aligned}$$

Let us denote the roots of (13) by $\gamma_A$ and $\gamma_B$

$$\gamma_A,\ \gamma_B = \frac{-2V_2 \pm 2\sqrt{V_2^2 - V_1(V_4 - 1/\cos^2\theta_0)}}{V_1}$$

The distances $\{d_j\}$ needed in equation (10) are defined as

$$2d_j = \psi_A(\lambda_j) + \psi_B(\lambda_j) \qquad j = 1 \dots n-1$$

The two angles $\psi_A$ and $\psi_B \in [0 \dots \pi/2]$, expressed in radians, are equal to the length of the arc lying on the unity sphere and linking the projected ellipsoid center $\alpha a$ and the projected point $x$ determined by (12). If $< x, y >= \sum_{i=1}^{n} x_i y_i$ represents the scalar product in the Euclidean space $\mathbb{R}^n$, then

$$cos(\psi_A) =< \frac{x(\lambda_j, \gamma_A)}{\|x(\lambda_j, \gamma_A)\|}, \frac{\alpha a}{\|\alpha a\|} > \qquad cos(\psi_B) =< \frac{x(\lambda_j, \gamma_B)}{\|x(\lambda_j, \gamma_B)\|}, \frac{\alpha a}{\|\alpha a\|} >$$

To summarize, for a given fading $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, evaluate the equivalent solid angle $\Omega(\theta_1)$ using (10), find $\theta_1$ by solving (3) and then compute the conditional error probability $Q(\theta_1)$ from equation (7).

# 5 Numerical results

Notice that the optimal performance analysis, i.e. the lower bound $Q(\theta_0)$ of section 3 and the approximation $Q(\theta_1)$ of section 4, give an estimation of the optimal word error probability versus the signal-to-noise ratio. In order to estimate the bit error probability, we consider

$$P_{eb} \approx \frac{d_{Hmin}}{n} P_{ew}$$

where $d_{Hmin}$ is the minimum Hamming distance of the code. Gallager [4] showed that when $n$ is large enough, binary random codes quickly approach the Gilbert-Varshamov bound, i.e.

$$H_2\Big(\delta = \frac{d_{Hmin}}{n}\Big) = 1 - R$$

where $H_2(x) = -x \log x - (1-x)\log(1-x)$ is the entropy function, and $0 < \delta \leq \frac{1}{2}$. Consequently

$$P_{eb} \approx H^{-1}_{2|[0,1/2]}(1-R) \times P_{ew} \tag{14}$$

Typical values of $\delta$ are listed in the table below

| R | $\delta = H^{-1}_{2\|[0,1/2]}(1-R)$ |
|---|---|
| 1/4 | 0.2145 |
| 1/2 | 0.1100 |
| 3/4 | 0.0417 |

The bit error rate performance of an optimal code is compared to that of a parallel concatenated convolutional code (Turbo code [2]) in figure 3. The code rate is 0.5 bits per dimension. The turbo code constituent is the recursive systematic convolutional code defined by the octal generators 23, 35. The number of decoding iterations is 12 for the length $n = 2048$ (interleaver size 1024) and 20 decoding iterations for the length $n = 131072$ (interleaver size 65536). As illustrated, the turbo code is 0.9 dB away from the optimal code at $10^{-5}$ when $n = 2048$. The gap is about 0.6 dB at $10^{-4}$ for the longer code $n = 131072$.

Figure 4 illustrates the performance of a parallel turbo code on a Rayleigh fading channel. The code parameters are identical to the one used in figure 3. The turbo code is 0.45 dB away from the optimal code at $10^{-6}$.

Finally, figures 5, 6 & 7 show the optimal bit error probability for a binary code of rate 1/4, 1/2 and 3/4 respectively. The curves are plotted for four different lengths $n = 100$, 200, 800, 2000 bits on both Gaussian and Rayleigh fading channels. The curves on the fading channel exhibit a very high diversity since the minimum Hamming distance of the random code derived from the Gilbert-Varshamov bound is relatively large. For the greatest code length $n = 2000$, the gap between the Rayleigh and the Gaussian channels is 1.4, 2.3 & 2.5 dB with the code rates 0.25, 0.50 & 0.75 bits/dim respectively.

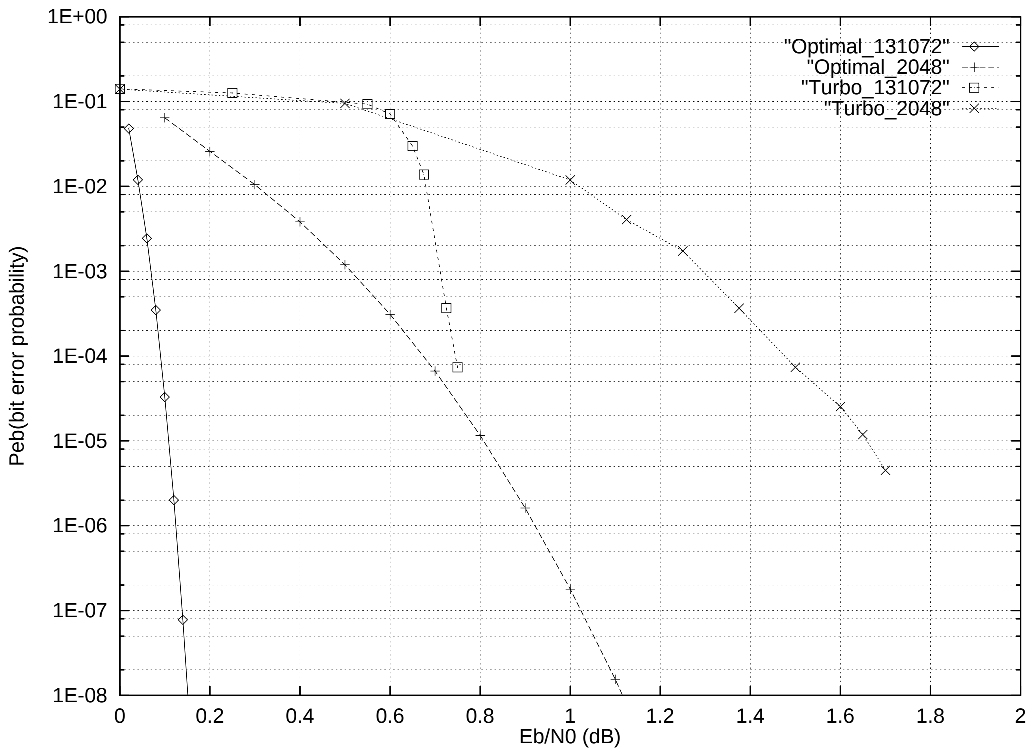


Figure 3: Turbo codes versus optimal codes, $n = 131072$ & 2048, rate 1/2, AWGN

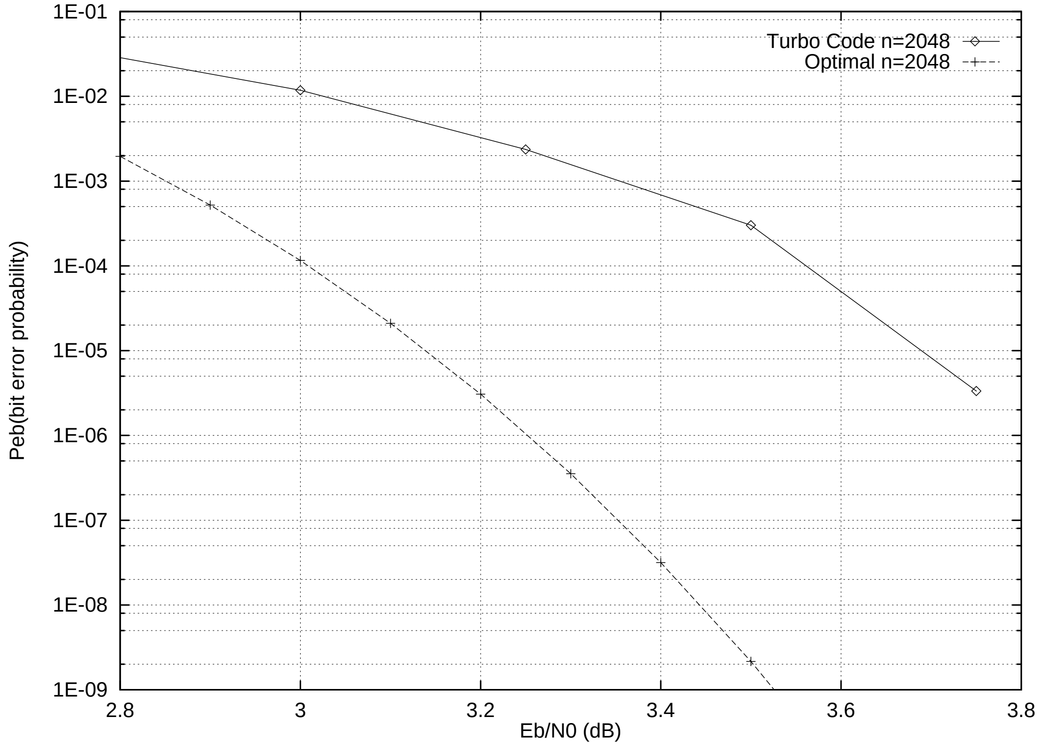


Figure 4: Turbo code versus optimal code, $n = 2048$, rate 1/2, Rayleigh

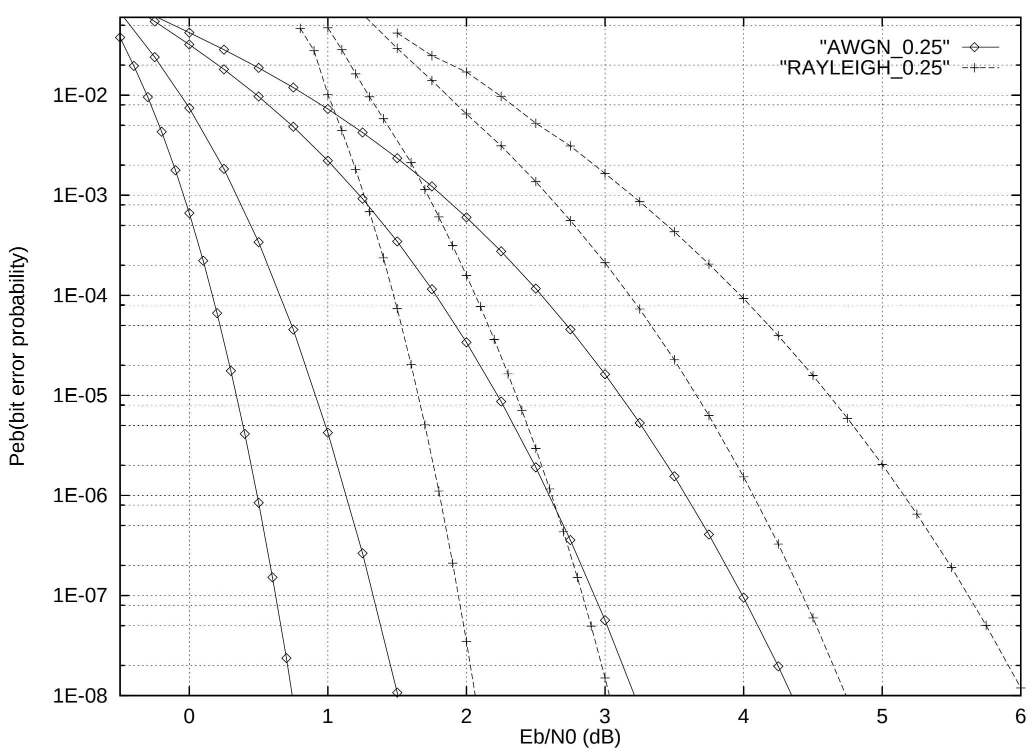


Figure 5: Optimal performance, rate 1/4, code length $n = 100, 200, 800, 2000$

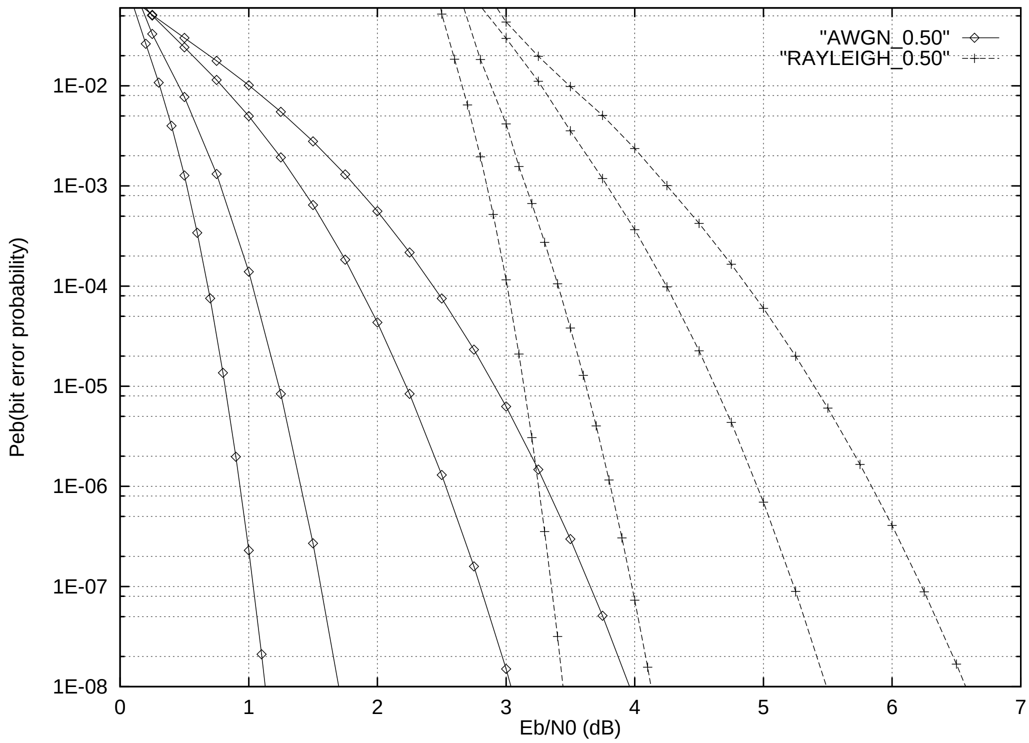


Figure 6: Optimal performance, rate 1/2, code length $n = 100, 200, 800, 2000$

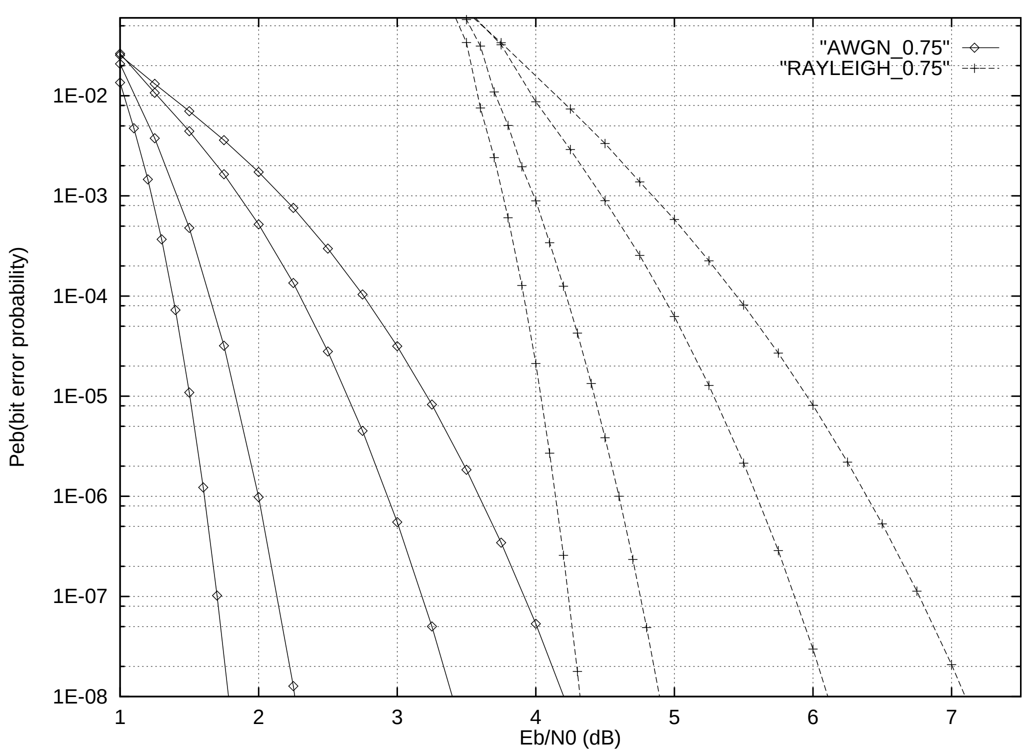


Figure 7: Optimal performance, rate 3/4, code length $n = 100, 200, 800, 2000$

# References


[1] G. Battail, H. Magalhaes de Oliveira : "Probabilité d'erreur du codage aléatoire avec décodage optimal sur le canal à bruit gaussien additif affecté ou non de fluctuations d'amplitude,", *Annales des Télécommunications* vol.48, no 1-2, pp. 15-28, 1993.

[2] C. Berrou, A. Glavieux, P. Thitimajshima: "Near Shannon Limit Error-Correcting Coding and Decoding: Turbo-Codes," *Proc. of ICC'93, Geneva* pp. 1064-1070, 23-26 May 1993.

[3] J.H. Conway, N.J. Sloane, *Sphere packings, lattices and groups*, 3rd edition, 1998, Springer-Verlag, New York.

[4] R.G. Gallager: Low-density parity-check codes, MIT Press, 1963.

[5] D.E. Lazic, T. Beth, M. Calic: "How close are turbo codes to optimal codes?," *Proc. Int. Symp. on Turbo Codes*, pp. 192–195, Brest, September 1997.

[6] John G. Proakis, *Digital Communications*, 3rd edition, McGraw-Hill, 1995.

[7] C.E. Shannon: "Probability of Error for Optimal Code in Gaussian Channel," *The Bell System Technical Journal*, vol. 38, no.3, May 1959.

[8] A.J. Viterbi, J.K. Omura: Principles of Digital Communication and Coding, McGraw-Hill Book Company, 1979.